Joint decision making and cooperative solutions

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Orandum est ut sit mens sana in corpore sano

Juvenal, *Satires*

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> I'm a lucky man to count on both hands
> the ones I love.
>
> Pearl Jam, *Just Breathe*

To get a good impression of the years that lead to this dissertation, it is best to take a few steps back and look at it from a distance. Since I decided to go out West (where the wind blows tall!), my distance to Tilburg got pretty large. This, together with a personal habit to appreciate things even more once they're behind me, makes me think this is a good time to reflect on the past years. I'll start with a small synopsis: thank you, it's been great!

The first of many thanks in this chapter should go out to the one that deserves it most: Peter Borm. Including the second year project 'zelfstandig modelleren', this is my fifth thesis supervised by Peter. He taught me the basics of research, to be careful before saying 'but that's straightforward'. But most importantly he introduced me to game theory and sparked my enthusiasm for the subject. For the Research Master's thesis, Peter brought in Marco Slikker as a second supervisor and he stayed on board for the Ph. D. Thesis. I couldn't have wished for a better supervising team. It is great how the two of you were able to guide me in my research, motivate me whenever my enthusiasm was fading and help me out in the ongoing battle to write everything down in a mathematically concise way.

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This leads me to the third part of the job as a Ph. D. student: the annual conferences. The basic idea here is to go to some foreign city to present and discuss your work, to meet fellow researchers and to get inspiration for future research. Fortunately, it is easy to add a few days and enjoy the city and the weather a bit with a few colleagues. Gerwald, John, Marloes, Mirjam, Soesja and Ruud: thanks for the good company.

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Edwin Lohmann

Devon, November 2011

Contents

Chapter 1

Introduction

Game theory is a branch of mathematics, describing and analyzing the interaction among decision makers. Where decision theory comprises the problem of finding a preferred alternative when only one decision maker is present, game theory assumes the existence of more than one decision maker while the actions of one decision maker possibly have an effect on the outcome for others. Central in game theory are the conflict situations arising from this interaction. These conflict situations can be of diverse nature, e.g., the fundamental conflict situation of warfare, the evolution of species in biology, voting in politics and profit sharing in economics.

The publication by Von Neumann (1928) laid the groundwork for game theory, but not before the book by Von Neumann and Morgenstern (1944) game theory attracted the attention of a wider audience. Generally, game theory is divided into two main parts: non-cooperative game theory and cooperative game theory.

In non-cooperative game theory, the involved parties, or *players*, act in self-interest and the competitive nature of interaction is dominant. A focus is on finding rational outcomes, equilibria: those strategy combinations where none of the players can improve by unilaterally changing his strategy. Ex ante the players cannot make binding agreements.

In cooperative game theory, the main topic of this dissertation, the players are assumed to cooperate to achieve an outcome preferred by the group as a whole and the conflict arises from dividing the jointly achieved payoff stemming from this outcome. So, the focus is on sharing the profit obtained from cooperation. The type of cooperative games considered in this thesis, are transferable utility games. In this framework, players are allowed to transfer utility, such as money, from one player to another. Contrary to the non-cooperative setup there is a mechanism, such as a legal system, allowing the players ex ante to make binding agreements on dividing the joint payoff. An important modeling aspect is an adequate translation from the conflict situation to a cooperative game, in which one explicitly defines the capabilities of subgroups of the players. A solution concept assigns an allocation to each game in a class of transferable utility games. To divide the joint profits, multiple solution concepts have been developed to capture different notions of the idea of a ‘fair’ allocation. A central notion is the core, the set of those allocations for which no group of players has an incentive to split off.

A specific area of application, as discussed in this thesis, is joint decision making within the field of operations research. Operations research deals with optimization problems in business and industry, often with a combinatorial aspect. When different players are involved in the decision making process or in the execution of tasks, we enter the area of operations research games.

This thesis can roughly be divided into three parts. After the preliminaries, Chapter 3 and 4 of this dissertation contain contributions to the theory of transferable utility games in general. Then, the next two chapters analyze specific cooperative situations originating from operations research problems. The last chapter is somewhat of an outlier as it does not involve transferable utility games but provides a taxonomy of rankings in tournaments.

As mentioned before, in cooperative game theory a number of solution concepts attracted attention. The Shapley value, the nucleolus and the compromise value are three of these well-studied solution concepts. For games with a non-empty core Chapter 3, which is based on Tijs et al. (2011), introduces an alternative to these concepts called the Alexia value. This solution concept is defined via so-called lexinals. Given an ordering on the set of players, the corresponding lexinal is such that every player takes the maximum he can obtain, respecting the restrictions of the core and the amounts already allocated to his predecessors. Subsequently, to obtain the Alexia value one averages all lexinals.

The Alexia value is analyzed for several classes of games. Since the Alexia value depends on the game through its core only, we focus on classes of games for which the core has a nice structure. A key concept in this analysis is compromise stability: for subclasses of the class of compromise stable games such as strongly compromise admissible games and big boss games, we show that the Alexia value coincides with the nucleolus. Also, we relate the Alexia value to concepts originating from the theory of bankruptcy situations.

A transferable utility game is called exact if there exists for every coalition, an allocation in the core such that this coalition receives exactly their value in the game. To check for exactness of a game, Csóka et al. (2011) introduced the equivalent concept of exact balancedness. Chapter 4, which is based on Lohmann et al. (2011), introduces minimal exact balanced collections as those exact balanced collections, for which no proper subset exists that is also exact balanced. We show that all other exact balanced collections are redundant for determining exactness of the game. This significantly reduces the number of conditions to be checked for exactness. An exact balanced collection not containing the grand coalition is minimal exact if and only if the corresponding exact balanced weight vector is unique. On the other hand, an exact balanced collection containing the grand coalition is minimal exact if and only if all corresponding exact balanced weight vectors impose an equivalent condition on the game.

Furthermore, it is shown that the class of minimal exact balanced collections can be partitioned into three types. The first type is the class of minimal balanced collections. The second type is formed by those collections that can be obtained from a minimal balanced collection by replacing one coalition, with a weight strictly smaller than one, by its complement. Finally, the third type is formed by all minimal balanced collections for every proper subgame, to which two coalitions are added: the grand coalition of the subgame, and the grand coalition of the original game.

Chapter 5, which is based on Tejada et al. (2011), discusses a general framework for games derived from a non-negative, square matrix in which every entry represents the value obtained from combining the corresponding row and column. We assume that every row and every column is associated with a player, where every player is associated to at most one row and at most one column simultaneously. In the special case that every player is associated with one row or one column only, the model boils down to the assignment problem (cf. Shapley and Shubik (1972)), in the special case that every player is associated with both a column and a row, the model corresponds to the permutation problem (cf. Tijs et al. (1984)). Within the general framework, we can associate every assignment problem to a permutation problem. We show how all extreme points of the core of the related permutation game can be viewed as extreme points of the core of the underlying assignment game. Although in general not all extreme points of the underlying assignment game are covered in this way, we prove that this is the case within the special class of Bohm-Bawerk assignment games.

In the last part of Chapter 5 the attention is shifted to permutation situations and games only. We study the structure of the set of all matrices that lead to permutations games with the same core. Moreover, we study a specific subclass of permutation situations called 'homogeneous alternatives' permutation situations in which the value obtained by a player while combining his row with the column of another player is independent of the column player.

In sequencing situations a number of jobs have to be processed on one or more machines, in such a way that a cost criterion is minimized. For one-machine sequencing situations where the cost incurred by a job is a linear function of its completion time, the order minimizing total costs is such that the jobs are processed in a non-increasing order with respect to their urgency (cost per time unit divided by the length of the job, cf. Smith (1956)). Curiel, Pederzoli, and Tijs (1989) initiated the game theoretic study of this type of operations research games.

Chapter 6, which is based on Lohmann et al. (2010), extends the literature on sequencing games. We introduce the model of sequencing situations with Just-in-Time (JiT) arrival. Comparing with the standard sequencing model, this model has two distinct features. First, instead of waiting in a queue from the moment the first job starts, a job arrives at the factory as soon as its predecessor is finished. Second, we introduce a setup time: in between jobs, the machine should be adjusted for the next job. The setup time we introduce in our model is taken to depend on the predecessor only, so the time between finishing a job and the start of the processing of the next job, does not depend on the job that is processed next. We restrict ourselves to those situations with JiT arrival, where two values for the setup time and two values for the cost parameter are allowed.

We discuss sequencing situations with JiT arrival from two perspectives. First, from an operations research perspective we solve the optimization problem regarding the joint cost of all players. Second, we use the setting of cooperative game theory to analyze the problem of dividing the costs of the optimal order among the players. We show that the core of a JiT sequencing game is always non-empty, and for large classes of JiT sequencing games we provide explicit expressions for both the core and the nucleolus. Also, we introduce a JiT sequencing specific allocation rule that provides a core element.

Chapter 7, which is based on González-Díaz et al. (2011), considers the problem of ranking a number of alternatives on the basis of information on pairwise comparisons of the alternatives. The set of alternatives along with the matrix containing the, possibly partial, information on the pairwise comparisons is called a tournament. Tournaments are discussed in the literature on a variety of subjects such as statistics, psychology, social choice and voting. Whereas the literature on tournaments usually assumes that the result of a pairwise comparison is binary, we assume that the result of a comparison between two alternatives is a pair of non-negative reals adding up to one, representing the result of both alternatives in the comparison. Hence, the model is able to capture more general measures of relative strength.

For this general setup, we provide a taxonomy of (the natural extensions of) common ranking methods for tournaments in the literature, and a new ranking method called recursive Buchholz on the basis of properties. We use terminology and interpretation from sports competitions for expositional purposes, our results are context free. In the process, we provide an adaptation of the characterization of the fair bets ranking method in Slutzki and Volij (2005) to our setting.

Chapter 2

Preliminaries

In these preliminaries, we introduce the basic mathematical notation that is used throughout this dissertation. Also, we introduce cooperative games and the main solution concepts.

2.1 Basic mathematical notation

Throughout this dissertation, N denotes a finite player set. We denote by 2^N the powerset of N, *i.e.*, the collection of all subsets of N. Also, $\mathcal{N} = 2^N \backslash \{\emptyset\}$ denotes the collection of non-empty subsets of N. For every $S \in \mathcal{N}$, the indicator vector $e^S \in \mathbb{R}^N$ is such that $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \in N \backslash S$. An *order* σ of N is a bijective function $\sigma : \{1, ..., |N|\} \to N$, where $\sigma(k)$ denotes the player at position $k \in \{1, ..., |N|\}$ in the order σ. The set of all orders of N is denoted with $\Pi(N)$. Given $\sigma \in \Pi(N)$, we will use the notation $\overline{\sigma}$ for the reverse order: $\overline{\sigma}(1) = \sigma(|N|)$, $\overline{\sigma}(2) = \sigma(|N| - 1)$,..., $\overline{\sigma}(|N|) = \sigma(1)$.

Given a polytope $P \subseteq \mathbb{R}^n$, a vector $x \in P$ is an extreme point if there are no $x', x'' \in P$, $x' \neq x''$ such that $x = \frac{1}{2}x' + \frac{1}{2}x''$. For a polytope P, let $\text{ext}(P)$ denote the set of extreme points, let $\text{int}(P)$ denote the relative interior and let $\dim(P)$ denote the dimension. For a polytope P of dimension n, a *facet* is an $(n-1)$-dimensional face. The *Euclidean distance* between two points $x, y \in \mathbb{R}^n$ is given by $d(x, y) = \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2}$. For $x \in \mathbb{R}^n$, the *open ball* with center x and radius ε is given by $B_\varepsilon(x) = \{y \in \mathbb{R}^n \mid d(x, y) < \varepsilon\}$.

2.2 Cooperative game theory

A *transferable utility (TU) game* (N, v) is defined by a finite player set N and a function v on the set 2^N of all subsets of N assigning to each coalition $S \in 2^N$ a value $v(S)$, describing the profit for the players in S when this coalition is formed. By convention, $v(\emptyset) = 0$. The class of TU games with player set N is denoted by TU^N. When no confusion can arise, we denote v for $(N, v) \in TU^N$.

An *allocation* is a vector $x \in \mathbb{R}^N$ where for every $i \in N$, x_i denotes the worth allocated to player i. For the game $v \in TU^N$, an allocation $x \in \mathbb{R}^N$ is called *efficient* if $\sum_{i \in N} x_i = v(N)$.

The *imputation set* $I(v)$ of a game $v \in TU^N$ is given by all individually rational and efficient allocations, so

$$I(v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), x_i \geq v(\{i\}) \text{ for all } i \in N\}.$$

For a game $v \in TU^N$ the *core* $C(v)$ (Gillies (1953)) is defined as the set of those efficient allocations of $v(N)$, for which no coalition has an incentive to split off:

$$C(v) = \Big\{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in \mathcal{N}\Big\}.$$

A game is called *balanced* if its core is non-empty (cf. Bondareva (1963), Shapley (1967)). The class of balanced games with player set N is denoted by Γ^N. For a game (N, v) and coalition $S \in 2^N$, the subgame (S, v_S) is given by $v_S(T) = v(T)$ for every $T \in 2^N$. A game $v \in TU^N$ is called *totally balanced* if the core of every subgame is non-empty.

For a game $v \in TU^N$ the *utopia vector* $M(v) \in \mathbb{R}^N$ is given by:

$$M_i(v) = v(N) - v(N \backslash \{i\}),$$

for every $i \in N$, and the *vector of minimum rights* is given by:

$$m_i(v) = \max_{S: i \in S} \Big\{v(S) - \sum_{j \in S \backslash \{i\}} M_j(v)\Big\},$$

for all $i \in N$. For a game $v \in TU^N$ the *core-cover* $CC(v)$ (Tijs and Lipperts (1982)) is defined as the set of those efficient allocations of $v(N)$, where every player receives at least his minimum right and at most his utopia demand:

$$CC(v) = \Big\{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N),\ m(v) \le x \le M(v)\Big\},$$

It is readily verified that $C(v) \subseteq CC(v)$. For a game $v \in TU^N$, the *Weber set* (Weber (1988)) is the convex hull of all marginal vectors:

$$W(v) = \text{conv}\{m^\sigma(v) \mid \sigma \in \Pi(N)\},$$

where for an order $\sigma \in \Pi(N)$, the *marginal vector* $m^\sigma(v) \in \mathbb{R}^N$ is formed by the marginal contributions with respect to σ:

$$m^\sigma_{\sigma(k)}(v) = v(\{\sigma(1), \sigma(2), ..., \sigma(k)\}) - v(\{\sigma(1), \sigma(2), ..., \sigma(k-1)\}),$$

for every $k \in \{1, ..., |N|\}$.

A game $v \in TU^N$ is *additive* if for some $a \in \mathbb{R}^N$, $v(S) = \sum_{i \in S} a_i$ for all $S \in \mathcal{N}$. A game $v \in TU^N$ is called *convex* (Shapley (1971)) if

$$v(S \cup \{i\}) - v(S) \le v(T \cup \{i\}) - v(T),$$

for all $i \in N$, $S \subseteq T \subseteq N \backslash \{i\}$. Convex games are those games for which the Weber set and the core coincide (cf. Shapley (1971), Ichiishi (1981)).

A *solution concept* defined on $\Sigma \subseteq TU^N$ assigns an allocation to each game $v \in \Sigma$. So, a solution concept could be defined on a subset of all TU-games only. Note that we define solution concepts as being single-valued. Throughout this dissertation the Shapley value, the compromise value and the nucleolus are often used solution concepts.

For a game $v \in TU^N$ the *Shapley value* $\Phi(v)$ (Shapley (1953)) is defined by

$$\Phi(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^\sigma(v).$$

We define the *excess* of coalition $S \in \mathcal{N}$ with respect to allocation $x \in I(v)$ by $E(S, x) = v(S) - \sum_{i \in S} x_i$. The excess measures the dissatisfaction of coalition S with respect to allocation x. Let $\omega(x) \in \mathbb{R}^{2^{|N|}}$ be the vector of excesses of $x \in I(v)$, arranged in weakly decreasing order. For $x, y \in \mathbb{R}^n$, x is lexicographically smaller than y, denoted by $x \leq_L y$, if $x = y$ or if there exists some $k \in \{1, ..., n\}$ such that $x_i = y_i$ for every $i \in \{1, ..., k-1\}$ and $x_k < y_k$.

For each game (N, v) such that $I(v) \neq \emptyset$, the *nucleolus* $\eta(v)$ (Schmeidler (1969)) is defined as the unique allocation $x \in I(v)$ such that $\omega(x) \leq_L \omega(y)$ for every $y \in I(v)$. For every game $v \in \Gamma^N$ we have $\eta(v) \in C(v)$.

For a game $v \in TU^N$ such that $CC(v) \neq \emptyset$, the *compromise value* $\tau(v)$ (Tijs (1981)) is the unique efficient combination of the utopia vector and the vector of minimum rights. For every $i \in N$:

$$\tau_i(v) = aM(v) + (1-a)m(v),$$

where $a \in [0, 1]$ is such that $\sum_{i \in N} \tau_i(v) = v(N)$.

We introduce a number of standard properties for solution concepts: efficiency, relative invariance with respect to strategic equivalence, symmetry and dummy. Let ψ be a solution concept defined on $\Sigma \subseteq TU^N$.

The solution concept ψ satisfies *efficiency* on the domain Σ if $\sum_{i \in N} \psi_i(v) = v(N)$ for every $v \in \Sigma$.

Two games $v \in TU^N$ and $w \in TU^N$ are *strategically equivalent* (Tijs (1976)) if there exist a vector $a \in \mathbb{R}^N$ and a positive real number k such that $w(S) = kv(S) + \sum_{i \in S} a_i$ for every $S \in 2^N$. The solution concept ψ satisfies *relative invariance with respect to strategic equivalence* on the domain Σ if $\psi(w) = k\psi(v) + a$ for all $v, w \in \Sigma$ such that $w(S) = kv(S) + \sum_{i \in S} a_i$ for every $S \in 2^N$ and for some positive real number k and vector $a \in \mathbb{R}^N$.

Two players $i \in N$ and $j \in N$ are *symmetric* in the game $v \in TU^N$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for every $S \subseteq N \backslash \{i, j\}$. The solution concept ψ satisfies *symmetry* on the domain Σ if, for all $v \in \Sigma$, $\psi_i(v) = \psi_j(v)$ for every $i \in N$ and $j \in N$ that are symmetric in v.

A player $i \in N$ is a *dummy* in the game $v \in TU^N$ if $v(S \cup \{i\}) - v(S) = v(\{i\})$ for every $S \subseteq N \backslash \{i\}$. The solution concept ψ satisfies the *dummy property* on the domain Σ if, for all $v \in \Sigma$, $\psi_i(v) = v(\{i\})$ for every $i \in N$ that is a dummy player in v.

Chapter 3

The Alexia value

3.1 Introduction

This chapter, which is based on Tijs et al. (2011), introduces the Alexia value, a new core selector defined for cooperative games with a non-empty core. Established one-point solution concepts such as the Shapley value for general TU-games and the compromise value for compromise admissible TU-games, in general do not provide a core-element. Defined for TU-games that allow for efficient and individually rational allocations, the nucleolus provides a core-element for all games with a non-empty core. In this respect, the Alexia value provides an alternative to the nucleolus.

The idea underlying the Alexia value is inspired by assignment problems, where a number of objects are to be allocated among an equal number of agents. One of the methods used to assign objects to agents is the so-called serial dictatorship as discussed by e.g. Abdulkadiroğlu and Sönmez (1998) and Bogomolnaia and Moulin (2001). Given an ordering over the players, this mechanism assigns the first player in the order his top choice, the second player is assigned his top choice among the remaining objects and so on. Of course, this method discriminates between the agents, and to restore fairness the order is chosen randomly. As we are dealing with transferable utility rather than objects, we can use the serial dictatorship to obtain an allocation for every order of players, and average these allocations. We can put numerous restrictions on the set of basic allocations to choose from. If we use the Weber set - the convex hull of all marginal vectors - as this set of basic allocations, this procedure would lead to the Shapley value. In this chapter we will concentrate on the basic set of efficient and coalitionally stable allocations, the core. A similar approach is also used in defining the 'run-to-the-bank rule' (O'Neill

(1982)) for bankruptcy situations.

This chapter defines the Alexia value via so-called lexinals. Here a lexinal is defined as a lexicographical maximum of the core, with respect to an arbitrary order on the players. Subsequently, to obtain the Alexia value one averages all lexinals. The Alexia value can be seen as a 'run-to-the-core rule' for games with a non-empty core, as for every lexinal players are running to the core according to a certain order. Every player then takes the maximum he can obtain within the subset of the core that remains after the players before him have made their respective choices.

This way, the Alexia value combines two often applied arguments with respect to choosing an allocation: using orderings on the players, while at the same time respecting the fairness criterion of the core. Hence, it combines attractive properties of widely accepted allocation rules such as the Shapley value and the nucleolus and, in fact, will coincide with these solution concepts for several classes of games. An application area of particular interest is formed by the class of operations research games. These games are used to divide cost savings stemming from operations research problems with several decision makers. The before mentioned properties as well as its intuitive nature make the Alexia value an attractive allocation rule: using, e.g., the application area of flow games, we show that the Alexia value can provide an appealing alternative to both the nucleolus and the Shapley value.

Since the Alexia value is based on the core rather than on the game itself, it is interesting to analyze classes of games where the core has a nice structure. For convex games, those games where the marginal contribution of a player increases if he joins a larger coalition, the Alexia value and the Shapley value coincide. Also, compromise stability turns out to be an important notion with respect to identifying the Alexia value. A game is called compromise stable (Quant et al. (2005)) if it has a non-empty core, and the core coincides with the core cover. Firstly, we show that for the class of strongly compromise admissible games à la Driessen (1988), which form a specific subclass of compromise stable games (for which there is a high incentive to cooperate and form the grand coalition), the Alexia value and the nucleolus coincide. Secondly, we discuss the Alexia value of clan games, which also form a subclass of compromise stable games. Clan games (Potters et al. (1989)) are games with a nonempty coalition called the clan, of which every member has veto-power, and where it is more profitable for the other players to unite in the negotiations than to form smaller coalitions. For clan games, an explicit expression for the Alexia value is derived. For big boss games (Muto et al. (1988)), the subclass of clan games for which the clan consists of one player only, the Alexia value again coincides with the nucleolus. For any game with a non-empty core, the exactification is defined as the unique exact game with the same core. Hence, by definition the Alexia value of a game with a non-empty core and of its exactification coincide. We use the exactification to show that the Alexia value of a compromise stable game coincides with the compromise extension of the run-to-the-bank rule for bankruptcy situations as introduced by Quant et al. (2006).

Finally, we introduce the reverse Alexia value, which averages over the lexicographic minima of the core. This approach can be seen as dual to the Alexia value: whereas the Alexia assumes that every player takes his restricted maximum, the reverse Alexia assumes that every player is sent away with the minimum that has to be allocated to him, respecting the restrictions of the core and the amount already allocated to his predecessors. We show that for compromise stable games and convex games the Alexia value coincides with the reverse Alexia value.

The outline of this chapter is as follows. Section 3.2 introduces the Alexia value and discusses some basic properties. Section 3.3 contains the results on the Alexia value on specific classes of games such as convex games, strongly compromise admissible games and big boss games. Section 3.4 analyzes the reverse Alexia value and we discuss other possible modifications.

3.2 The Alexia value

To define the Alexia value, we first introduce the notion of a lexinal. Also, for the lexinals and for the Alexia value we provide a number of basic results. We show that every lexinal is an extreme point of the core, but that there can exist extreme points that are not a lexinal. We discuss the Alexia value on 2-person games and state several standard properties for solution concepts that are satisfied by the Alexia value.

Definition 3.2.1 For $(N, v) \in \Gamma^N$ and an order $\sigma \in \Pi(N)$, the *lexinal* $\lambda^\sigma(v) \in \mathbb{R}^N$ is defined as the lexicographic maximum on $C(v)$ with respect to σ, i.e.,

$$\lambda^\sigma_{\sigma(k)}(v) = \max\Big\{x_{\sigma(k)} \mid x \in C(v),\ x_{\sigma(l)} = \lambda^\sigma_{\sigma(l)}(v) \text{ for all } l \in \{1, .., k-1\}\Big\},$$

for all $k \in \{1, ..., |N|\}$.

A lexinal is recursively defined such that every player gets the maximum he can obtain inside the core under the restriction that the players before him in the corresponding order obtain their restricted maxima. It is readily checked that every lexinal is an extreme point of the core.

Theorem 3.2.2 Let $(N, v) \in \Gamma^N$. Then for every $\sigma \in \Pi(N)$, $\lambda^\sigma(v)$ is an extreme point of the core.

Proof: Let $\sigma \in \Pi(N)$. By construction, $\lambda^\sigma(v) \in C(v)$. Assume that $\sigma \in \Pi(N)$ is such that $\lambda^\sigma(v)$ is not an extreme point of the core. Then there exist $x^1, x^2 \in C(v)$ with $x^1 \neq x^2$ such that $\lambda^\sigma(v) = ax^1 + (1-a)x^2$ for some $a \in (0,1)$. Take $l \in \{1, ..., |N|\}$ such that $x^1_{\sigma(k)} = x^2_{\sigma(k)}$ for every $k \in \{1, ..., l-1\}$ and $x^1_{\sigma(l)} \neq x^2_{\sigma(l)}$. Without loss of generality we can assume $x^1_{\sigma(l)} > x^2_{\sigma(l)}$. This means that

$$\max\{x_{\sigma(k)} \mid x \in C(v), x_{\sigma(l)} = \lambda^\sigma_{\sigma(l)}(v) \text{ for all } l \in \{1, .., k-1\}\} \geq x^1_{\sigma(l)} > \lambda^\sigma_{\sigma(l)},$$

which contradicts the definition of a lexinal. Hence, $\lambda^\sigma(v)$ is an extreme point of $C(v)$. $\square$

However, for some games there exist extreme points of the core that do not coincide with a lexinal, which is shown in the next example. The game we consider is a variant of an example in Derks and Kuipers (2002).

Example 3.2.3 Let (N, v) be the game with $N = \{1, 2, 3, 4\}$, $v(\{1\}) = \frac{1}{2}$, $v(\{2\}) = v(\{3\}) = v(\{4\}) = 0$, and

$$v(S) = \begin{cases} 7 & \text{if } |S| = 2, \\ 12 & \text{if } |S| = 3, \\ 22 & \text{if } S = N. \end{cases}$$

For this example, we demonstrate how one can compute the core. In the remainder of this dissertation, we will omit the details of computing the core. Every core-element $x \in C(v)$ satisfies the following system of (in)equalities:

$$\begin{aligned} x_1 &\geq \tfrac{1}{2} & \qquad x_2 + x_4 &\geq 7 \\ x_2 &\geq 0 & x_3 + x_4 &\geq 7 \\ x_3 &\geq 0 & x_1 + x_2 + x_3 &\geq 12 \\ x_4 &\geq 0 & x_1 + x_2 + x_4 &\geq 12 \\ x_1 + x_2 &\geq 7 & x_1 + x_3 + x_4 &\geq 12 \\ x_1 + x_3 &\geq 7 & x_2 + x_3 + x_4 &\geq 12 \\ x_1 + x_4 &\geq 7 & x_1 + x_2 + x_3 + x_4 &= 22 \\ x_2 + x_3 &\geq 7 & & \end{aligned}$$

A core-element $x \in C(v)$ is an extreme point if in the above system 4 linearly independent (in)equalities hold with equality. As $x_1 + x_2 + x_3 + x_4 = 22$ holds for every $x \in C(v)$, we have at most $\binom{14}{3} = 364$ candidate extreme points. For each combination of three inequalities of the above system, we can check if the equalities corresponding to these inequalities are linearly independent and check if the resulting allocation is a core element. If both conditions hold, we obtained an extreme point of the core. E.g., the equalities corresponding with $x_2 \geq 0$, $x_1 \geq \frac{1}{2}$ and $x_1 + x_2 \geq 7$ are linearly dependent and therefore do not define an extreme point. If the inequalities $x_3 \geq 0$, $x_4 \geq 0$ and $x_2 + x_4 \geq 7$ all hold with equality, we obtain the allocation $x = (15, 7, 0, 0)$. However, this is not a core element, as $x_2 + x_3 + x_4 = 7 < 12$. Now, if the inequalities $x_1 \geq \frac{1}{2}$, $x_1 + x_2 \geq 7$ and $x_1 + x_4 \geq 7$ (for which the corresponding equalities are linearly independent) all hold with equality, we obtain $x = (\frac{1}{2}, 6\frac{1}{2}, 8\frac{1}{2}, 6\frac{1}{2})$. As this allocation satisfies the above system of (in)equalities, x is an extreme point of the core. Checking all combinations results in the following 24 extreme points of the core:

(i) 12 extreme points which are permutations of $(10, 5, 5, 2)$,

(ii) 9 extreme points which are permutations of $(7, 7, 8, 0)$ but with first coordinate unequal to 0,

(iii) $(\frac{1}{2}, 6\frac{1}{2}, 6\frac{1}{2}, 8\frac{1}{2})$, $(\frac{1}{2}, 6\frac{1}{2}, 8\frac{1}{2}, 6\frac{1}{2})$ and $(\frac{1}{2}, 8\frac{1}{2}, 6\frac{1}{2}, 6\frac{1}{2})$.

Let $\sigma \in \Pi(N)$. We have $\lambda^{\sigma}_{\sigma(1)}(v) = \max\{x_{\sigma(1)} \mid x \in C(v)\} = 10$. As every lexinal is an extreme point of the core, $\lambda^{\sigma}(v)$ is equal to a permutation of $(10, 5, 5, 2)$. So, the extreme points given by *(ii)* and *(iii)* are not lexinals. $\triangleleft$

Now we can define the Alexia value using the notion of a lexinal.

Definition 3.2.4 For $(N,v) \in \Gamma^N$, the *Alexia value* $\alpha(v)$ is defined as the average over the lexinals:

$$\alpha(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} \lambda^\sigma(v).$$

Since every lexinal is a core element, the average over all lexinals is also a core element. Hence, for every $(N,v) \in \Gamma^N$, $\alpha(v) \in C(v)$.

The Alexia value combines attractive properties of both the nucleolus and the Shapley value. First of all, just as the nucleolus, the Alexia value is a core-selector which guarantees coalitional stability. Furthermore, just as the Shapley value averages over the marginal vector of every ordering, the Alexia value starts out by selecting a core allocation for every ordering and averages over all these allocations.

The following example demonstrates the Alexia value, and shows that it coincides with the standard solution for 2-person games.

Example 3.2.5 Let $(N,v) \in \Gamma^N$ with $N = \{1,2\}$. Then $v(N) \geq v(\{1\}) + v(\{2\})$ and $C(v)$ =conv$\{f^1, f^2\}$ with $f^1 = (v(N) - v(\{2\}), v(\{2\}))$, $f^2 = (v(\{1\}), v(N) - v(\{1\}))$. Clearly for the lexinals one finds $\lambda^{(1,2)}(v) = f^1$ and $\lambda^{(2,1)}(v) = f^2$. So, the Alexia value $\alpha(v) = \frac{1}{2}(f^1 + f^2)$ equals

$$\left(v(\{1\}) + \tfrac{1}{2}\Big(v(N) - v(\{1\}) - v(\{2\})\Big), v(\{2\}) + \frac{1}{2}\Big(v(N) - v(\{1\}) - v(\{2\})\Big)\right),$$

the standard solution for the 2-person game (N,v). ◁

We show that the Alexia value satisfies a number of standard properties for solution concepts.

Theorem 3.2.6 The Alexia value satisfies efficiency, relative invariance w.r.t. strategic equivalence, symmetry and dummy on Γ^N.

Proof:
Efficiency. Let $(N,v) \in \Gamma^N$ and $\sigma \in \Pi(N)$. It holds that $\sum_{i\in N} \lambda_i^\sigma(v) = v(N)$ as $\sum_{i \in N} x_i = v(N)$ for every $x \in C(v)$ and $\lambda^\sigma(v) \in C(v)$. Therefore, $\sum_{i\in N} \alpha_i(v) = \frac{1}{|N|!} \sum_{\sigma\in\Pi(N)} \lambda^\sigma(v) = v(N)$.

Relative invariance w.r.t. strategic equivalence. Let $(N, v) \in \Gamma^N$ and $(N, w) \in \Gamma^N$ be strategic equivalent. Take an additive game (N, a) and $k \in \mathbb{R}_{++}$ such that $w = kv+a$. Since $x \in C(v)$ if and only if $kx+a \in C(w)$, we have that $\lambda^\sigma(w) = k\lambda^\sigma(v)+a$ for every $\sigma \in \Pi(N)$ and therefore $\alpha(w) = k\alpha(v) + a$. So, the Alexia value satisfies relative invariance with respect to strategic equivalence.
Symmetry. Let $(N, v) \in \Gamma^N$, $i \in N$ and $j \in N$ be such that i and j are symmetric in (N, v). Take $x \in C(v)$ and let $\overline{x}$ be such that $\overline{x}_h = x_h$ for every $h \in N\backslash\{i, j\}$, $\overline{x}_i = x_j$ and $\overline{x}_j = x_i$. Then $\overline{x} \in C(v)$. Hence, for every $\sigma \in \Pi(N)$ we have $\lambda_i^\sigma(v) = \lambda_j^{\sigma'}(v)$, where $\sigma'(h) = \sigma(h)$ for every $h \in N\backslash\{i, j\}$, $\sigma'(i) = \sigma(j)$ and $\sigma'(j) = \sigma(i)$. Therefore, $\alpha_i(v) = \alpha_j(v)$ and we obtain that the Alexia value satisfies symmetry.
Dummy. Let $(N, v) \in \Gamma^N$ and $i \in N$ be such that i is a dummy player in (N, v). As for every $x \in C(v)$ we have $x_i = v(\{i\})$, we obtain $\lambda_i^\sigma(v) = v(\{i\})$ for every $\sigma \in \Pi(N)$. Hence, $\alpha_i(v) = v(\{i\})$ so the Alexia value satisfies the dummy property. □

The properties mentioned in the theorem above are satisfied by, e.g., the Shapley value as well. A difference between these solution concepts can be found in the following characterizations. A solution concept ψ defined on the domain $\Sigma \subseteq TU^N$ satisfies *balanced average contributions* on Σ if, for any $(N, v) \in \Sigma$ with $|N| \geq 2$ and any $i \in N$ it holds that

$$\frac{1}{|N|-1} \sum_{j \in N\backslash\{i\}} (\psi_i(v) - \psi_i(v_{N\backslash\{j\}})) = \frac{1}{|N|-1} \sum_{j \in N\backslash\{i\}} (\psi_j(v) - \psi_j(v_{N\backslash\{i\}})).$$

The value $\psi_i(v) - \psi_i(v_{N\backslash\{j\}})$ can be seen as the contribution of player $j \in N$ to player $i \in N$, as the allocation to player i increases by $\psi_i(v) - \psi_i(v_{N\backslash\{j\}})$ because of the presence of player j. The property of balanced average contributions says that the average contribution of player $i \in N$ to the other players equals the average contribution of the other players to player i. Kongo et al. (2010) uses balanced average contributions to characterize the Shapley value.

Theorem 3.2.7 (Kongo et al. (2010)) The Shapley value is the unique solution concept on TU^N that satisfies efficiency and balanced average contributions.

The Alexia value can be characterized by similar properties. Let $(N, v) \in \Gamma^N$ with $|N| \geq 2$. Define $A^i(v) = \max\{x_i \mid x \in C(v)\}$ as the maximum payoff in the core for player $i \in N$, and define the Davis-Maschler (DM) reduced game $(N\backslash\{i\}, v^{-i})$ by

$$v^{-i}(S) = \max\{v(S), v(S \cup \{i\}) - A^i(v)\},$$

for all $S \subseteq N \backslash \{i\}$. Hence, if (N, v) is convex then $(N \backslash \{i\}, v^{-i})$ is the subgame of v with respect to player set $N \backslash \{i\}$.

The solution concept ψ satisfies the *balanced average DM-contributions* on the domain $\Sigma \subseteq \Gamma^N$ if, for all $v \in \Sigma$ with $|N| \geq 2$ and any $i \in N$,

$$\frac{1}{|N|-1} \sum_{j \in N \backslash \{i\}} (\psi_i(v) - \psi_i(v^{-j})) = \frac{1}{|N|-1} \sum_{j \in N \backslash \{i\}} (\psi_j(v) - \psi_j(v^{-i})).$$

This property states that for every player $i \in N$, the average difference between his payoff in the original game and his payoff in the game where another player has an advantageous position equals the average difference between the payoff of the other players in the original game and their payoff in the game where player i has an advantageous position. This property can be used to characterize the Alexia value.

Theorem 3.2.8 (Kongo et al. (2010)) The Alexia value is the unique solution concept which satisfies balanced average DM-contributions and efficiency on the domain Γ^N.

So, the Shapley value and the Alexia value are similar in the sense that both solution concepts balance the average amount a player contributes to another, but differ in the way these contributions are measured. For the Shapley value, a player is sent away with his marginal contribution when forming the grand coalition, whereas for the Alexia value this player obtains his maximum pay-off in the core of the game.

The DM reduced game can also be used to provide an alternative expression for the Alexia value.

Proposition 3.2.9 *(cf. Caprari et al. (2008)) Let $(N, v) \in \Gamma^N$. Then*

$$\alpha_i(v) = \frac{1}{|N|}(A^i(v) + \sum_{j \in N \backslash \{i\}} \alpha_i(v^{-j})),$$

for all $i \in N$.

Proposition 3.2.9 states that the Alexia value can be computed by averaging over $|N|$ allocations. In each of these allocations one player $j \in N$ is assigned his maximum core payoff $A^j(v)$, and the remainder $v(N) - A^j(v)$ is allocated according to the Alexia value of an appropriately defined game v^{-j} on the remaining players in $N \backslash \{j\}$. Note that if $C(v) \neq \emptyset$ then also $C(v^{-j}) \neq \emptyset$, so the $\alpha(v^{-j})$ is indeed defined.

3.3 The Alexia value on classes of games

For several classes of games, the Alexia value coincides with either the Shapley value or the nucleolus. First, we discuss the class of convex games, where the Shapley value equals the Alexia value. Secondly, we focus on two subclasses of the class of compromise stable games: for strongly compromise admissible games and big boss games the Alexia value coincides with the nucleolus. For clan games and simple flow games we provide explicit expressions for the Alexia value. Finally, we use exactification to show that the Alexia value equals the compromise extension of the run-to-the-bank rule.

First of all, the differences between the Alexia value and both the Shapley value and the nucleolus are showed. The following example shows positive features of the Alexia value and its advantages over the nucleolus and the Shapley value in the application area of flow games as introduced by Kalai and Zemel (1982). We follow the slightly different definition of Granot and Granot (1992).

To describe a *flow network* f we first need an undirected graph $G = (V, E)$. The set of vertices V contains two distinguished vertices: the source (So) and the Sink (Si). A flow network is further described by a capacity function: every edge $e \in E$ has a nonnegative capacity $\mathrm{cap}(e) \in \mathbb{R}_+$. Lastly, every edge $e \in E$ is owned by a coalition of players $S(e) \in \mathcal{N}$. From the flow network f, we obtain the flow game v_f by defining the value $v_f(S)$ of coalition $S \in 2^N$ as the amount that S can transport from source to sink, while utilizing only edges that are owned by the players in S.

For simple flow networks, it is assumed that every player owns exactly one edge and every edge has capacity 1.

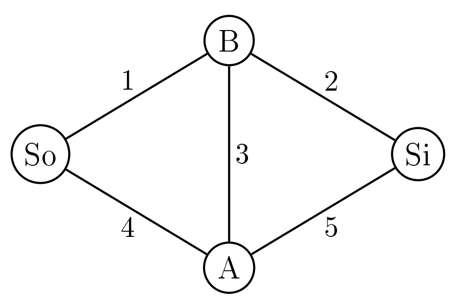

Figure 3.3.1: The simple flow network f

Example 3.3.1 In the simple flow network of Figure 3.3.1, all edges have capacity 1 and are undirected. The players want to generate a flow from the source So to the sink Si. The number next to an edge denotes the player that owns the edge: e.g. player 4 owns the edge $\{So, A\}$. The player set is $N = \{1, ..., 5\}$. For every coalition $S \subseteq N$ it can be computed how much they can transport per time unit from the source to the sink without using edges owned by players outside the coalition. This leads to the value $v_f(S)$ of the coalition S in the flow game (N, v_f) corresponding with the simple flow network. One readily checks that the core of this game equals conv$\{(1,0,0,1,0),(0,1,0,0,1)\}$, while the Alexia value (and also the nucleolus) is given by $\alpha(v_f) = (\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2})$. Note that the Shapley value $\Phi(v_f) = (\frac{29}{60}, \frac{29}{60}, \frac{1}{15}, \frac{29}{60}, \frac{29}{60})$ lies outside the core. Next, consider the flow network g (see Figure

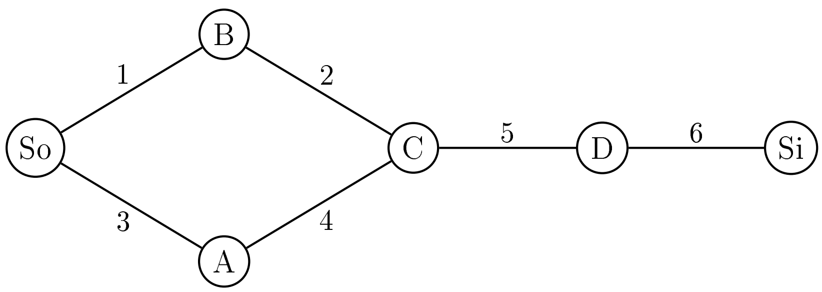

Figure 3.3.2: The flow network g

), where all edges have capacity 1 except edges $\{C, D\}$ and $\{D, Si\}$ which have capacity 2. This means that e.g. $v_g(\{1,2,5,6\}) = v_g(\{3,4,5,6\}) = 1$ and $v_g(N) = 2$ for the corresponding flow game v_g. Consider the allocation $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The highest excess of x is $-\frac{1}{3}$ which is attained at coalitions $\{1,2,5,6\}, \{3,4,5,6\}$ and at all one-person coalitions. The vector of excesses of this allocation is lexicographically smaller than the vector excesses of any other element of the imputation set, as for every $x \in I(v)$, $x_i < \frac{1}{3}$ for some $i \in N$. Hence, $\eta(v_g) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, while $\alpha(v_g) = \Phi(v_g) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2})$. The nucleolus allocates $v_g(N)$ equally among all players, although one could argue that both the position in the graph and the capacity of the edges of player 5 and 6 are superior to those of the other players. This discrepancy between the players is reflected in the Alexia value. $\triangleleft$

The result in the first part of Example 3.3.1 can be generalized. For a flow network f, a cut is a coalition $S \in 2^N$ such that no positive flow can be generated from source to sink without using at least one edge owned by S. So, $S \in 2^N$ is a cut in the flow network f if $v_f(N \backslash S) = 0$. By Υ^f we denote the set of cuts in the flow network f. Define the set of minimum cuts $\Upsilon^f_{min} = \{S \in \Upsilon^f \mid |S| \leq |T| \text{ for every } T \in \Upsilon^f\}$. By Reijnierse et al. (1996), for a simple flow network f we have $C(v_f) = \text{conv}\{e^S \mid S \in \Upsilon^f_{min}\}$. As every minimum cut vector consists of the same number of ones and zeros, every minimum cut vector occurs the same number of times as lexinal. Therefore, $\alpha(v_f)$ equals the average over all minimal cut vectors.

Proposition 3.3.2 *Let f be a simple flow network. Then*

$$\alpha(v_f) = \frac{1}{|\Upsilon^f_{min}|} \sum_{S \in \Upsilon^f_{min}} e^S.$$

3.3.1 Convex games

For the class of convex games, the Alexia coincides with the Shapley value.

Theorem 3.3.3 Let (N, v) be convex. Then $\alpha(v) = \Phi(v)$.

Proof: Since (N, v) is convex, $C(v) = \text{conv}\{m^\sigma(v) \mid \sigma \in \Pi(N)\}$ and, by Theorem 3.2.2 , every lexinal is some marginal vector. In fact, we will show that $\lambda^\sigma(v) = m^{\overline{\sigma}}(v)$ for all $\sigma \in \Pi(N)$. Let $\sigma \in \Pi(N)$. By convexity, $v(N) - v(N \backslash \{i\}) \geq v(S) - v(S \backslash \{i\})$ for all $S \subseteq N \backslash \{i\}$, and hence $\lambda^\sigma_{\sigma(1)}(v) = \max_{x \in C(v)} x_{\sigma(1)} = m^{\overline{\sigma}}_{\sigma(1)}(v)$ using convexity. Take $k \in \{2, ..., |N|\}$. Now assume that $\lambda^\sigma_{\sigma(j)}(v) = m^{\overline{\sigma}}_{\sigma(j)}(v)$ for all $j \in \{1, ..., k-1\}$. Take $S = \{\sigma(k+1), ..., \sigma(|N|)\}$. Then $\sum_{j=1}^{k-1} \lambda^\sigma_{\sigma(j)}(v) = v(N) - v(S \cup \{\sigma(k)\})$. Since $\lambda^\sigma(v) \in C(v)$, it must hold that $\sum_{i \in S} \lambda^\sigma_i(v) \geq v(S)$. Hence, $\lambda^\sigma_{\sigma(k)}(v) \leq v(S \cup \{\sigma(k)\}) - v(S) = m^{\overline{\sigma}}_{\sigma(k)}(v)$. But as $m^{\overline{\sigma}}(v) \in C(v)$ we know $\lambda^\sigma_{\sigma(k)}(v) = m^{\overline{\sigma}}_{\sigma(k)}(v)$. It now follows that $\alpha(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} \lambda^\sigma(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^{\overline{\sigma}}(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^\sigma(v) = \Phi(v)$. □

The following theorem shows that the Alexia value is additive on the class of convex games. Together with Theorem 3.3.3, this provides an alternative proof for the additivity of the Shapley value on the class of convex games.

Theorem 3.3.4 Let (N, v) and (N, w) be convex games. Then $\alpha(v + w) = \alpha(v) + \alpha(w)$.

Proof: We have $C(v)+C(w)=C(v+w)$, since $C(v)=W(v)$, $C(w)=W(w)$ and

$$\begin{aligned} m^{\sigma}_{\sigma(k)}(v+w) &= v(\{\sigma(1),...,\sigma(k)\})+w(\{\sigma(1),...,\sigma(k)\}) \\ &\quad -v(\{\sigma(1),...,\sigma(k-1)\})-w(\{\sigma(1),...,\sigma(k-1)\}) \\ &= m^{\sigma}_{\sigma(k)}(v)+m^{\sigma}_{\sigma(k)}(w), \end{aligned}$$

for every $k \in \{1,...,|N|\}$. This means that $\alpha(v+w)=\alpha(v)+\alpha(w)$, since $\lambda^{\sigma}(v+w)=m^{\overline{\sigma}}(v+w)=m^{\overline{\sigma}}(v)+m^{\overline{\sigma}}(w)=\lambda^{\sigma}(v)+\lambda^{\sigma}(w)$ for every $\sigma \in \Pi(N)$. □

We now turn to several subclasses of the class of compromise stable games, such as strongly compromise admissible games and clan games.

3.3.2 Strongly compromise admissible games

A game is called *compromise admissible* (also known as quasi-balanced) if the core cover is non-empty. For a compromise admissible game (N,v) one can characterize the core cover with the use of larginals. For all $\sigma \in \Pi(N)$, the *larginal* $\ell^{\sigma}(v)$ is defined by

$$\ell^{\sigma}_{\sigma(k)}(v)=\begin{cases} M_{\sigma(k)}(v) \quad \text{if} \quad \sum_{j=1}^{k} M_{\sigma(j)}(v)+\sum_{j=k+1}^{|N|} m_{\sigma(j)}(v) \leq v(N), \\ m_{\sigma(k)}(v) \quad \text{if} \quad \sum_{j=1}^{k-1} M_{\sigma(j)}(v)+\sum_{j=k}^{|N|} m_{\sigma(j)}(v) \geq v(N), \\ v(N)-\sum_{j=1}^{k-1} M_{\sigma(j)}(v)-\sum_{j=k+1}^{|N|} m_{\sigma(j)}(v) \text{ otherwise,} \end{cases}$$

for all $k \in \{1,\ldots,|N|\}$. For each compromise admissible game (N,v) the core cover coincides with the convex hull of all larginal vectors:

$$CC(v)=\text{conv}\{\ell^{\sigma}(v) \mid \sigma \in \Pi(N)\}.$$

A compromise admissible game (N,v) is called *compromise stable* (Quant et al. (2005)) if the core cover equals the core.

A compromise admissible game (N,v) is called *strongly compromise admissible* (cf. Driessen (1988)) if for all $S \in \mathcal{N}$ it holds that:

$$v(N) - v(S) \geq \sum_{i \in N \setminus S} M_i(v)$$

Strongly compromise admissible games are also called dual simplex games or 1-convex games. From Quant et al. (2005) it follows that every strongly compromise admissible game is compromise stable.

Note that compromise admissibility and strongly compromise admissibility imply opposite conditions on the utopia demands. Compromise admissibility implies that the utopia demands are large: there exists an efficient allocation such that every player obtains at most his utopia demand. Strongly compromise admissibility on the other hand implies that the utopia demands are small: if a coalition is assigned exactly its value, all players outside a coalition can obtain their utopia demands. In fact, Driessen (1988) shows that for every strongly compromise admissible game (N, v) it holds that $v(N) = \sum_{j \in N \setminus \{i\}} M_j(v) + m_i(v)$ for every $i \in N$.

For our result on the Alexia value of strongly compromise admissible games, we use the following expressions for the core and the nucleolus of strongly compromise admissible games, derived by Driessen (1988).

Theorem 3.3.5 (cf. Driessen (1988)) If (N, v) is strongly compromise admissible, then

(i) $C(v) = \text{conv}\left\{\{M(v)e^{N \setminus \{i\}} + m(v)e^{\{i\}}\}_{i \in N}\right\}$,

(ii) $\eta(v) = \tau(v) = \frac{|N|-1}{|N|} M(v) + \frac{1}{|N|} m(v)$, the barycenter of C(v).

For strongly compromise admissible games, the Alexia value coincides with the nucleolus.

Theorem 3.3.6 Let (N, v) be strongly compromise admissible. Then $\alpha(v) = \eta(v) = \tau(v)$.

Proof: Let (N, v) be strongly compromise admissible. By part (i) of Theorem 3.3.5, $C(v) = \text{conv}\left\{\{M(v)e^{N \setminus \{i\}} + m(v)e^{\{i\}}\}_{i \in N}\right\}$. Let $\sigma \in \Pi(N)$. Then $\lambda^{\sigma}_{\sigma(k)}(v) = M_{\sigma(k)}(v)$ for all $k \in \{1, ..., |N| - 1\}$, and $\lambda^{\sigma}_{\sigma(|N|)}(v) = m_{\sigma(|N|)}(v)$. Therefore,

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[0]As an allocation rule, the barycenter of the core is known as the core-center. For a study on the core-center, see e.g. González-Díaz and Sánchez-Rodríguez (2007) and González-Díaz and Sánchez-Rodríguez (2009)

$$\begin{aligned}\alpha(v) &= \frac{1}{|N|!}\sum_{\sigma\in\Pi(N)}\lambda^{\sigma}(v)\\ &= \frac{1}{|N|!}(|N|!-(|N|-1)!)M(v)+((|N|-1)!)m(v)]\\ &= \eta(v)\\ &= \tau(v),\end{aligned}$$

where the last equalities follow from Theorem 3.3.5 (*ii*). □

3.3.3 Clan games

Clan games are introduced in Potters et al. (1989). Big boss games, which form a subclass of clan games, are introduced in Muto et al. (1988) and are further considered in Branzei and Tijs (2001) and Tijs and Branzei (2002). For big boss games, we show that the Alexia value coincides with the nucleolus. This does not hold for the more general class of clan games, but we obtain an explicit expression for the Alexia value on this class of games.

A game (N, v) is called a *clan game* if $v(S) \geq 0$ for all $S \in 2^N$, $M(v) \geq 0$ and if there exists a coalition $C \in \mathcal{N}$ called the clan such that:

(i) Clan property: $v(S) = 0$ for all S with $C \not\subseteq S$.

(ii) Union property: $v(N) - v(S) \geq \sum_{i\in N\setminus S} M_i(v)$ for each $S \in 2^N$ with $C \subseteq S$.

Note that all players in C are symmetric in (N, v) as $v(S) = 0$ for every $S \in 2^N$, $C \not\subseteq S$. A clan game (N, v) is called a *big boss game* if there exists a clan $C = \{b\}$ for some $b \in N$. The player b is called the *big boss*.

As is shown by Quant et al. (2005), clan games are compromise stable. However they are not necessarily strongly compromise admissible.

Example 3.3.7 Consider the clan game (N, v), defined by $N = \{1, 2, 3\}$ and

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v(S)$	0	0	0	5	0	0	10

This game is a clan game with $C = \{1, 2\}$. Clearly, $M(v) = (10, 10, 5)$. This game is not strongly compromise admissible, since $v(N) - v(\{1\}) = 10 < 15 = \sum_{j\in N\setminus\{1\}} M_j(v)$. ◁

Potters et al. (1989) provide an explicit expression for the core of a clan game. For a clan game (N, v) with clan $C \in \mathcal{N}$, $S \subseteq N \backslash C$ and $i \in C$ define $z^{S,i} \in \mathbb{R}^N$ by

$$z_j^{S,i} = \begin{cases} M_j(v) & \text{if } j \in S, \\ v(N) - \sum\limits_{h \in S} M_h(v) & \text{if } j = i, \\ 0 & \text{else,} \end{cases}$$

for all $j \in N$.

Theorem 3.3.8

(i) *(cf. Potters et al. (1989)).* Let (N, v) be a clan game with clan $C \in \mathcal{N}$. Then

$$C(v) = \text{conv}\{z^{S,i} \mid S \subseteq N \backslash C,\ i \in C\}.$$

(ii) *(cf. Muto et al. (1988)).* Let (N, v) be a big boss game with big boss $b \in N$. Then, for all $i \in N$,

$$\eta_i(v) = \begin{cases} \frac{1}{2} M_i(v) & \text{if } i \neq b, \\ v(N) - \dfrac{1}{2} \sum\limits_{j \in N \backslash \{b\}} M_j(v) & \text{if } i = b. \end{cases}$$

Theorem 3.3.9 Let (N, v) be a big boss game. Then $\alpha(v) = \eta(v)$.

Proof: Let player $b \in N$ be the big boss. By Theorem 3.3.8 (i) with $C = \{b\}$, for all $\sigma \in \Pi(N)$ and every $k \in \{1, ..., |N|\}$ such that $\sigma(k) \neq b$ it holds that

$$\lambda^{\sigma}_{\sigma(k)}(v) = \begin{cases} M_{\sigma(k)}(v) & \text{if } k < \sigma^{-1}(b), \\ 0 & \text{if } k > \sigma^{-1}(b). \end{cases}$$

Therefore, $\alpha_i(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} \lambda^{\sigma}_i(v) = \frac{1}{|N|!} M_i(v) \cdot |\{\sigma \in \Pi(N) | \sigma^{-1}(i) < \sigma^{-1}(b)\}| = \frac{1}{2} M_i(v) = \eta_i(v)$ for each $i \in N \backslash \{b\}$. By efficiency of η and α it then follows that $\alpha(v) = \eta(v)$. □

For clan games however, the Alexia value does not necessarily equal the nucleolus.

Example 3.3.10 Consider the clan game of Example 3.3.7. We have $\eta(v) = (3\frac{3}{4}, 3\frac{3}{4}, 2\frac{1}{2})$. The core is given by $C(v) = \text{conv}\{(5,0,5),(0,5,5),(10,0,0),(0,10,0)\}$. This implies that $\alpha(v) = \frac{1}{6} \cdot (25, 25, 10)$. ◁

The following theorem provides an explicit description of the Alexia value on the class of clan games.

Theorem 3.3.11 Let (N, v) be a clan game with clan $C \subseteq N$, $|C| \geq 2$. Then, for all $i \in N$,

$$\alpha_i(v) = \begin{cases} \frac{M_i(v)}{1+|C|} & \text{if } i \in N \backslash C, \\ \dfrac{v(N) - \sum_{j \in N \backslash C} \frac{M_j(v)}{1+|C|}}{|C|} & \text{if } i \in C. \end{cases}$$

Proof: By Theorem 3.3.8 (i), for all $j \notin C$ it holds that

$$\lambda_j^\sigma(v) = \begin{cases} M_j(v) & \text{if } \sigma^{-1}(j) < \sigma^{-1}(i) \text{ for all } i \in C, \\ 0 & \text{else.} \end{cases}$$

Let $j \in N \backslash C$. The number of orders where $\lambda_j^\sigma(v) = M_j(v)$ equals $\frac{|N|!}{|C|+1}$, since in a fraction $\frac{1}{|C|+1}$ of a total of $|N|!$ possible orders player j stands in front of all players in the clan. This implies that $\alpha_j(v) = \frac{M_j(v)}{|C|+1}$ for all $j \in N \backslash C$. Since the Alexia value is efficient, the remainder is divided among the clan members. As clan members are symmetric and the Alexia value satisfies symmetry, every clan member obtains an equal share of the remainder. □

3.3.4 Compromise stable games and exactification

To obtain an explicit expression for the Alexia value on the class of compromise stable games we use exactification. A balanced game (N, v) is called *exact* (Schmeidler (1972)) if for every coalition $S \subseteq N$ there exists an element $x \in C(v)$ such that $\sum_{i \in S} x_i = v(S)$. We refrain from a more elaborate discussion on the topic of exact games, as this is the topic of Chapter 4 of this dissertation. The *exactification* (N, v^E) of an arbitrary game $(N, v) \in \Gamma^N$ is the unique exact game with the same core as the original game v, i.e.,

$$v^E(S) = \min \Big\{ \sum_{i \in S} x_i \mid x \in C(v) \Big\},$$

for each $S \subseteq N$. So, $C(v^E) = C(v)$ for every $(N, v) \in \Gamma^N$ and $v^E = v$ if and only if (N, v) is exact. Note that if for two games (N, v) and (N, w) we have $C(v) = C(w) \neq \emptyset$, then $\alpha(v) = \alpha(w)$. Hence, α is *invariant with respect to exactification*: $\alpha(v) = \alpha(v^E)$ for every $(N, v) \in \Gamma^N$. We use exactification to show that for compromise stable games, the Alexia value coincides with the Shapley value of the exactification. Moreover, within this class of games the Alexia value equals the compromise extension of the run-to-the-bank rule for bankruptcy situations.

A *bankruptcy situation* is defined as a triple (N, E, d), where $E \in \mathbb{R}_+$ is the estate which has to be divided among a set of players N. The claim vector is denoted by $d \in \mathbb{R}^N_+$, where d_i represents the claim of player $i \in N$. It is assumed that $E \leq \sum_{i \in N} d_i$. With a bankruptcy situation one can associate a bankruptcy game $(N, v_{E,d})$, where the value of a coalition equals the amount of the estate not claimed by the players outside the coalition: $v_{E,d}(S) = \max\left\{0, E - \sum_{i \in N \setminus S} d_i\right\}$ for all $S \subseteq N$. Curiel et al. (1987) showed that every bankruptcy game is convex and compromise stable. O'Neill (1982) introduces the run-to-the-bank (RTB) rule to divide the estate among the claimants.

For a bankruptcy situation (N, E, d) the *run-to-the-bank rule* $RTB(E, d)$ is given by

$$RTB(E, d) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} r^\sigma(E, d),$$

where for all $\sigma \in \Pi(N)$ and $k \in \{1, ..., |N|\}$,

$$r^\sigma_{\sigma(k)}(E, d) = \max\left\{ \min\{d_{\sigma(k)}, E - \sum_{l=1}^{k-1} d_{\sigma(l)}\}, 0\right\}.$$

The interpretation of $r^\sigma_{\sigma(k)}$ is as follows: the players arrive at the bank according to the order σ. Upon arrival, a player receives his total claim or, if there is not enough money left to satisfy his claim, the maximum amount that is available. Importantly, for every bankruptcy situation (E, d) it holds that $RTB(E, d) = \Phi(v_{E,d})$.

The *compromise extension* RTB^* of the RTB-rule to the class of all compromise admissible games is introduced by Quant et al. (2006). For each compromise admissible game (N, v), the *compromise extension* RTB^* of the RTB-rule is given by

$$RTB^*(v) = m(v) + RTB(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)).$$

Theorem 3.3.12 Let (N, v) be compromise stable. Then

(i) (N, v^E) is strategically equivalent to a bankruptcy game.

(ii) $\alpha(v) = \Phi(v^E) = RTB^*(v)$.

Proof:
(i) Since (N, v) is compromise stable, $C(v) = CC(v)$. Hence, for all $S \subseteq N$

$$\begin{aligned} v^E(S) &= \min\{\sum_{i\in S} x_i \mid x \in C(v)\} \\ &= \min\{\sum_{i\in S} x_i \mid x \in CC(v)\} \\ &= \min\{\sum_{i\in S} \lambda_i^\sigma(v) \mid \sigma \in \Pi(N)\} \\ &= \sum_{i\in S} \lambda_i^{\sigma'}, \end{aligned}$$

where $\sigma' \in \Pi(N)$ is such that $\sigma'(k) \in N\backslash S$ for all $k \in \{1, ..., |N\backslash S|\}$. Therefore,

$$v^E(S) = \begin{cases} \sum_{i\in S} m_i(v) & \text{if } \sum_{i\in N\backslash S} M_i(v) + \sum_{i\in S} m_i(v) \geq v(N), \\ v(N) - \sum_{i\in N\backslash S} M_i(v) & \text{if } \sum_{i\in N\backslash S} M_i(v) + \sum_{i\in S} m_i(v) < v(N). \end{cases}$$

The first case corresponds with the pivot of $\lambda^{\sigma'}(v)$ (the first player in the order σ' that does not obtain his utopia demand) being a member of $N\backslash S$, and the second case corresponds with the pivot of $\lambda^{\sigma'}(v)$ being a member of S.
Consider the bankruptcy problem (N, E, d) with $E = v(N) - \sum_{i\in N} m_i(v)$ and $d_i = M_i(v) - m_i(v)$ for all $i \in N$. The corresponding bankruptcy game $(N, v_{E,d})$ is given by

$$v_{E,d}(S) = \max\Big\{0, v(N) - \sum_{i\in N\backslash S} M_i(v) - \sum_{i\in S} m_i(v)\Big\},$$

for all $S \in 2^N$.
Since $v^E = v_{E,d} + m(v)$, (N, v^E) is strategically equivalent to the game $(N, v_{E,d})$.

(ii) Because (N, v^E) is strategically equivalent to a bankruptcy game, (N, v^E) is convex and compromise stable. Since $C(v) = C(v^E)$, we have $\alpha(v) = \alpha(v^E)$. So, with $E = v(N) - \sum_{i \in N} m_i(v)$ and $d_i = M_i(v) - m_i(v)$ for all $i \in N$, the proof of (i) implies that

$$\begin{aligned} RTB^*(v) &= m(v) + RTB(v(N) - \sum_{i \in N} m(v), M(v) - m(v)) \\ &= m(v) + \Phi(v_{E,d}) \\ &= \Phi(v_{E,d} + m(v)), \end{aligned}$$

where the second equality holds as $RTB(E,d)$ and $\Phi(v_{E,d})$ coincide for every bankruptcy situation. The last equality is obtained by additivity of Φ: $\Phi(v) + \Phi(w) = \Phi(v + w)$ for every (N, v) and (N, w). Taking $w = v_{E,d}$ in the proof of part (i), we have

$$\begin{aligned} RTB^*(v) &= \Phi(v^E) \\ &= \alpha(v), \end{aligned}$$

where the last equality holds by Theorem 3.3.3. □

The previous theorem also provides an alternative proof for the fact that for two and three person games exactness is equivalent to convexity. For arbitrary N, convexity implies exactness as every marginal vector is an element of the core. Every exact game (N, v) is equal to its exactification (N, v^E). Since every balanced two or three person game is compromise stable, this implies that (N, v) is strategically equivalent with a bankruptcy game. As bankruptcy games are convex, we obtain that (N, v) is convex.

On the class of compromise stable games, which contains the class of bankruptcy games, one recognizes an essential difference between the Alexia value and the nucleolus.

To demonstrate this difference, we first introduce the Aumann-Maschler rule AM. Given a bankruptcy situation (N, E, d), denote $CEA(E, d)$ for the *constrained equal award rule*, given by

$$CEA_i(E, d) = \min\{a, d_i\},$$

for every $i \in N$, where $a \in \mathbb{R}_+$ is such that $\sum_{i \in N} \min\{a, d_i\} = E$.

Now *the Aumann-Maschler rule* $AM(E, d)$ (Aumann and Maschler (1985)) is given by

$$AM_i(E,d) = \begin{cases} CEA_i(E, \frac{1}{2}d) & \text{if } \sum_{j\in N} d_j \geq 2E \\ d_i - CEA_i(\sum_{j\in N} d_j - E, \frac{1}{2}d) & \text{if } \sum_{j\in N} d_j < 2E \end{cases}$$

for every $i \in N$.

For a compromise stable game (N, v), the Alexia value is given by:

$$\alpha(v) = m(v) + RTB(v(N) - \sum_{i\in N} m_i(v), M(v) - m(v)),$$

whereas Quant et al. (2005) show that

$$\eta(v) = m(v) + AM(v(N) - \sum_{i\in N} m_i(v), M(v) - m(v)).$$

So, the Alexia value and the nucleolus are similar in the sense that every player obtains his minimum right, and the framework of a bankruptcy situation is used to allocate the remainder. The difference between the Alexia value and the nucleolus lies in the treatment of this bankruptcy situation: the Alexia value uses the run-to-the-bank rule whereas the nucleolus uses the Aumann-Maschler rule to obtain an allocation for the related bankruptcy situation.

3.4 The reverse Alexia value and other modifications

The definition of the Alexia value allows for modifications in a number of different directions. Inspired by the literature on weighted Shapley values, Caprari et al. (2008) introduces two weighted Alexia values. These weighted Alexia values differ in the way the weights are assigned to orders: for the so-called μ-mixed lexicographic value weights are assigned to orders directly, whereas for the p-weighted lexicographic value weights are assigned to players and the p-weighted lexicographic value is the expectation of the allocation when players are sent away according to their relative weights. For every set of p-weights there exists a set of μ-weights such that the p-weighted lexicographic value and the μ-mixed lexicographic value coincide. Also, the p-weighted lexicographic value is extended to the (p, S)-weighted lexicographic value to circumvent the case where multiple players have zero p-weight. Caprari et al. (2008) generalizes a number of results mentioned in this chapter: for convex games the (p, S)-weighted lexicographic value coincides with the (p, S)-weighted Shapley value (Kalai and Samet (1988)) and for big boss games, clan games and strongly compromise admissible games (which are all compromise stable), the (p, S)-weighted lexicographic value coincides with the (p, S)-weighted Shapley value on the exactification of the game.

Here, we introduce another modification to the Alexia value. We introduce the reverse Alexia value, the average over all lexicographical minima of the core.

Definition 3.4.1 For $(N, v) \in \Gamma^N$, the reverse Alexia value $\overline{\alpha}(v)$ is given by

$$\overline{\alpha}(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} \overline{\lambda}^{\sigma}(v),$$

where for all $\sigma \in \Pi(N)$ the reverse lexinal $\overline{\lambda}^{\sigma}(v) \in \mathbb{R}^N$ is defined by

$$\overline{\lambda}^{\sigma}_{\sigma(k)}(v) = \min \left\{ x_{\sigma(k)} \mid \ x \in C(v),\ x_{\sigma(l)} = \overline{\lambda}^{\sigma}_{\sigma(l)}(v) \text{ for all } l \in \{1, ..., k-1\} \right\},$$

for all $k \in \{1, ..., |N|\}$.

Numerous results on lexinals and the Alexia value discussed before in this chapter also hold for reverse lexinals and the reverse Alexia value. Every reverse lexinal is an extreme point of the core, but for some games there exist extreme points of the core that do not coincide with a reverse lexinal. In Example 3.2.3, $(10, 5, 5, 2)$ is an extreme point of the core, but it is not a reverse lexinal. Using the same reasonings as for the Alexia value, it is obtained that the reverse Alexia value satisfies the properties of efficiency, relative invariance with respect to strategic equivalence, symmetry and dummy. As Example 3.4.4 shows that the Alexia value and the reverse Alexia value need not coincide, it follows that the reverse Alexia value does not satisfy balanced average DM-contributions.

It turns out that for compromise stable games, the Alexia value and the reverse Alexia value coincide.

Theorem 3.4.2 Let (N, v) be compromise stable. Then $\overline{\alpha}(v) = \alpha(v)$.

Proof: As (N, v) is compromise stable, $C(v) = CC(v) \neq \emptyset$. We showed that $\alpha(v) = RTB^*(v)$ and, by Quant et al. (2006), we have

$$RTB^*(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} \ell^{\sigma}(v).$$

Take $\sigma \in \Pi(N)$. We show that $\overline{\lambda}^{\sigma}(v) = \ell^{\overline{\sigma}}(v)$. Clearly,

$$\begin{aligned}\overline{\lambda}^{\sigma}_{\sigma(1)}(v) &= \min\left\{x_{\sigma(1)} \mid x \in C(v)\right\} \\ &= \min\left\{x_{\sigma(1)} \mid x \in CC(v)\right\}.\end{aligned}$$

Since $x_{\sigma(1)} \geq m_i(v)$ and $\sum_{i\in N\backslash\sigma(1)} x_i \leq \sum_{i\in N\backslash\{\sigma(1)\}} M_i(v)$, $\overline{\lambda}^{\sigma}_{\sigma(1)}(v) = \max\{m_{\sigma(1)}, v(N) - \sum_{i\in N\backslash\{\sigma(1)\}} M_i(v)\} = \ell^{\overline{\sigma}}_{\sigma(1)}(v)$. Now consider $k \in \{2, ..., |N|\}$ and assume that $\overline{\lambda}^{\sigma}_{\sigma(l)}(v) = \ell^{\overline{\sigma}}_{\sigma(l)}(v)$ for $l \in \{1, ..., k-1\}$. Take $S = \{\sigma(1), ..., \sigma(k)\}$. Then

$$\begin{aligned}\min\{\sum_{i\in S} x_i \mid x \in CC(v)\} &= \max\{v(N) - \sum_{i\in N\backslash S} M_i(v), \sum_{i\in S} m_i(v)\}, \\ &= \sum_{i\in S} \ell^{\overline{\sigma}}_i(v),\end{aligned}$$

where the first equality follows from the definition of the core-cover: $\sum_{i\in S} x_i \geq \sum_{i\in S} m_i(v)$ and $\sum_{i\in N\backslash S} x_i \leq \sum_{i\in N\backslash S} M_i(v)$. So, if $\overline{\lambda}^{\sigma}_{\sigma(l)}(v) = \ell^{\overline{\sigma}}_{\sigma(l)}(v)$ for $l \in \{1, ..., k-1\}$ then $\overline{\lambda}^{\sigma}_{\sigma(k)}(v) = \ell^{\overline{\sigma}}_{\sigma(k)}(v)$. By induction it follows that $\overline{\lambda}^{\sigma}(v) = \ell^{\overline{\sigma}}(v)$. This leads to

$$\overline{\alpha}(v) = \frac{1}{|N|!} \sum_{\sigma\in\Pi(N)} \ell^{\overline{\sigma}}(v) = \frac{1}{|N|!} \sum_{\sigma\in\Pi(N)} \ell^{\sigma}(v) = \alpha(v),$$

which concludes the proof. □

Note that the class of compromise stable games includes strongly compromise admissible games and clan games. Hence, most results derived in section 3 not only hold for the Alexia value, but for the reverse Alexia value as well. For convex games, the reverse Alexia value coincides with the Alexia value.

Theorem 3.4.3 Let (N, v) be convex. Then $\overline{\alpha}(v) = \alpha(v)$.

Proof: Let $\sigma \in \Pi(N)$. By induction on the position $k \in \{1, ..., |N|\}$, we prove that $\overline{\lambda}^{\sigma}(v) = m^{\sigma}(v)$. Consider the first player in the order:

$$\begin{aligned}\overline{\lambda}^{\sigma}_{\sigma(1)}(v) &= \min\left\{x_{\sigma(1)} \mid x \in C(v)\right\} \\ &= \min\left\{x_{\sigma(1)} \mid x \in W(v)\right\}\end{aligned}$$

Since $v(\{i\}) \leq v(S \cup \{i\}) - v(S)$ for every $S \in \mathcal{N}$, we have

$$\begin{aligned}\overline{\lambda}^{\sigma}_{\sigma(1)}(v) &= v(\{\sigma(1)\}) \\ &= m^{\sigma}_{\sigma(1)}(v).\end{aligned}$$

Now consider $k \in \{2, ..., |N|\}$ and, assume that $\overline{\lambda}^{\sigma}_{\sigma(l)}(v) = m^{\sigma}_{\sigma(l)}(v)$ for $l \in \{1, ..., k-1\}$. Then

$$\sum_{l=1}^{k-1} \overline{\lambda}^{\sigma}_{\sigma(l)}(v) = \sum_{l=1}^{k-1} m^{\sigma}_{\sigma(l)}(v) = v(\{\sigma(1), ..., \sigma(k-1)\}).$$

Let $S = \{\sigma(1), ..., \sigma(k)\}$. By definition of the core, $\min\left\{\sum_{i \in S} x_i \mid x \in C(v)\right\} \geq v(S)$. Therefore, $\overline{\lambda}^{\sigma}_{\sigma(k)}(v) \geq v(S) - v(S \backslash \{\sigma(k)\}) = m^{\sigma}_{\sigma(k)}(v)$. However, by convexity $C(v) = W(v)$. As $m^{\sigma}(v) \in W(v)$ we obtain $\overline{\lambda}^{\sigma}_{\sigma(k)}(v) = m^{\sigma}_{\sigma(k)}(v)$. We may conclude that $\overline{\lambda}^{\sigma}(v) = m^{\sigma}(v)$. Consequently,

$$\overline{\alpha}(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^{\sigma}(v) = \Phi(v).$$

By Theorem 3.3.3 we obtain $\overline{\alpha}(v) = \alpha(v)$. □

Note that both convexity and compromise stability imply that the core equals the convex hull of the lexinals. However, this property in itself does not guarantee the Alexia and the reverse Alexia value to coincide.

Example 3.4.4 Let (N, v) be given by $N = \{1, .., 4\}$ and

$$v(S) = \begin{cases} 15 & \text{if} \quad S = \{1\} \\ 0 & \text{if} \quad |S| = 1, S \neq \{1\} \text{ or } |S| = 3, \\ 8 & \text{if} \quad S = |2| \\ 30 & \text{if} \quad S = N \end{cases}$$

Then $C(v) =$conv$\{(15, 1, 7, 7), (15, 7, 1, 7), (15, 7, 7, 1), (18, 4, 4, 4)\}$. It is readily seen that every extreme point of the core equals a lexinal for 6 orders. However, $\alpha(v) = (15\frac{3}{4}, 4\frac{3}{4}, 4\frac{3}{4}, 4\frac{3}{4})$, while $\overline{\alpha}(v) = (15, 5, 5, 5)$ since $(18, 4, 4, 4)$ is not a reverse lexinal, and the other extreme points of the core equal a reverse lexinal for 8 orders each. ◁

Basically, the Alexia value is a run-to-the-core solution. Obviously one can modify the set the players run to. One could replace the core by any other set of solutions such as the Weber set, the core cover, a bargaining set or a share opportunity set as done in Caprari et al. (2006). The following example illustrates a bargaining set based Alexia value, with respect to the bargaining set of Aumann and Maschler (1964).

Example 3.4.5 Consider the five person glove game studied in Apartsin and Holzman (2003). Player 1 and 2 each own two right-hand gloves and player 3, 4 and 5 each own one left-hand glove. The value of a coalition is the number of pairs of gloves they can form. Obviously, the corresponding game (N, v) is given by $v(S) = \min\{\sum_{i\in S} b_1^i, \sum_{i\in S} b_2^i\}$ for all $S \in 2^N$, with b_1^i the number of right-hand gloves, and b_2^i the number of left-hand gloves of player $i \in N$. The core of this game consists of one point, $C(v) = \{(0, 0, 1, 1, 1,)\}$, so $\alpha(v) = (0, 0, 1, 1, 1)$. The Aumann-Maschler (A-M) bargaining set however is larger than the core, $\mathcal{M}(v) = \text{conv}\{(\frac{3}{2}, \frac{3}{2}, 0, 0, 0), (0, 0, 1, 1, 1)\}$. Let us consider the A-M bargaining set based Alexia value, which is the average of all lexicographic maxima of the A-M bargaining set. So, instead of considering core allocations, we consider only allocations in the A-M bargaining set. It is readily checked that for a fraction $\frac{2}{5}$ of all orders, the corresponding lexicographic maximum equals $(\frac{3}{2}, \frac{3}{2}, 0, 0, 0)$, and for the other orders the corresponding lexicographic maximum equals $(0, 0, 1, 1, 1)$. This leads to $(\frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5})$ for an A-M bargaining set based Alexia value. $\triangleleft$

Chapter 4

Minimal exact balancedness

4.1 Introduction

This chapter, which is based on Lohmann et al. (2011), studies the theoretical structure and properties of the class of exact balanced collections. A collection of coalitions is balanced if one can find positive weights for all coalitions in the collection such that every player is present in coalitions with total weight exactly equal to one. A game is balanced if for all such collections and all such weights, the weighted sum of the values of the coalitions does not exceed the value of the grand coalition. An interpretation is that the players can distribute one unit of working time among all coalitions in a way that for every coalition all members are active for an amount of time equal to the coalition's weight, and in doing so the players cannot create more value than by working one unit of time in the grand coalition. Bondareva (1963) and Shapley (1967) showed independently that balancedness is equivalent with non-emptiness of the core.

Exactness turns out to be equivalent with exact balancedness as introduced in Csóka et al. (2011). Exact balancedness is similar to the notion of balancedness, but we allow one of the weights to be negative. Classes of games that are exact are, e.g., convex games, risk allocation games with no aggregate uncertainty (Csóka et al. (2009)), convex multi-choice games (Branzei et al. (2009)) and multi-issue allocation games (Calleja et al. (2005)).

To verify that the core of a game is non-empty, not all balanced collections are needed. A balanced collection of coalitions is minimal, if there does not exist a proper subset that is also balanced. As it turns out, only minimal balanced collections have to be considered to ensure non-emptiness of the core. This greatly reduces the number of constraints to be checked for non-emptiness of the core. Furthermore, the class of minimal balanced collections is such that there exists no subclass of the class of minimal balanced collections that ensures balancedness of the game.

Regarding exact balancedness, many exact balanced collections are redundant when verifying the exactness of a game. This chapter provides a analysis of the class of minimal exact balanced collections: those exact balanced collections that do not contain a proper subset that is also exact balanced. We refrain from the computation aspects and focus on the theoretical structure of the class of minimal exact balanced collections. We show that only minimal exact balanced collections are essential to obtain exactness. However, it is not possible to use the same approach as with minimal balanced collections. This is due to the fact that while the set of balanced weight vectors is a convex set in which the extreme points are the weight vectors corresponding with minimal balanced collections, the set of exact balanced weight vectors is not a convex set. This requires a new approach for the proofs.

A main result shows that the class of minimal exact balanced collections can be partitioned into three types. The first type consists of all minimal balanced collections. The second type, the class of minimal negative balanced collections, consists of those collections that can be obtained from a minimal balanced collection by replacing one coalition, with a weight strictly smaller than one, by its complement. We show that for every minimal negative balanced collection there exists exactly one such combination of a minimal balanced collection and a coalition with a weight strictly smaller than one. The last type, the class of minimal subbalanced collections, is formed by all minimal balanced collections for every proper subset of the player set, to which two coalitions are added: the coalition formed by all players of the subset, and the coalition formed by all players of the original player set.

The class of minimal exact balanced collections ensures exactness of the game, but the class can be reduced even further. We show that only the class of minimal subbalanced collections and the class of minimal negative balanced collections are needed to guarantee exactness. So, the class of minimal balanced collections is redundant in this respect.

With respect to the uniqueness of the weights, it is well known that the class of minimal balanced collections coincides with the set of balanced collections for which the set of balanced weight vectors consists of one point. This result can be obtained partly for minimal exact balanced collections. If the exact balanced weight vector is unique for a certain exact balanced collection, then this collection is minimal exact balanced. The other way around however does not hold. For two types, minimal balanced and minimal negative balanced collections, the corresponding weight vector is unique. For every minimal subbalanced collection however, there exists more than one exact balanced weight vector. However, all weight vectors are related to each other by a linear transformation, and induce the same constraint on the game.

The chapter is organized as follows: the subsequent section introduces (minimal) balancedness and reviews the main results regarding balanced collections. Section 4.3 contains the definitions of several notions regarding exact balancedness, and includes the results on the uniqueness of the weights. Section 4.4 shows that the class of minimal exact balanced collections can be partitioned into three easily identifiable types. Section 4.5 states that minimal exact balanced collections are sufficient to ensure exactness of the game, and shows the redundancy of the minimal balanced collections in this respect.

4.2 Balancedness

First, we introduce balancedness and presents the main results on minimal balanced collections and balancedness. To check for non-emptiness of the core of a game (N, v), one can use the notion of balancedness.

Definition 4.2.1 Let $\mathcal{B} \subseteq \mathcal{N}, \mathcal{B} \neq \{N\}$. A weight vector $\beta \in \mathbb{R}^{\mathcal{N}}$ is called *balanced* on $\mathcal{B}$ if $\beta_S > 0$ for all $S \in \mathcal{B}$, $\beta_S = 0$ for all $S \notin \mathcal{B}$ and $\sum_{S \in \mathcal{B}} \beta_S e^S = e^N$. We denote $\Lambda^+(\mathcal{B})$ for the set of all balanced weight vectors on $\mathcal{B}$. The collection $\mathcal{B}$ is called *balanced* if $\Lambda^+(\mathcal{B}) \neq \emptyset$. Denote $\mathbb{B}^N$ for the set of all balanced collections on player set N, and $\Lambda^+ = \cup_{\mathcal{B} \in \mathbb{B}^N} \Lambda^+(\mathcal{B})$.

In the remainder, we will typically use $\mathcal{B}$ and $\mathcal{C}$ to denote balanced collections, and use β and γ to denote their respective weight vectors.

Example 4.2.2 Let $N = \{1, 2\}$. The collections $\{\{1\}\}$ and $\{\{2\}\}$ are not balanced, since one of the players is not present in the collection. By definition $\{\{1, 2\}\}$ is not balanced. The collection $\{\{1\}, \{1, 2\}\}$ is not balanced. This follows as a balanced weight vector β cannot satisfy the equations $\beta_{\{1,2\}} = 1$ and $\beta_{\{1\}} + \beta_{\{1,2\}} = 1$ simultaneously, since $\beta_{\{1\}} > 0$. A similar reasoning holds for the collection $\{\{2\}, \{1, 2\}\}$. The two remaining collections are $\mathcal{B} = \{\{1\}, \{2\}\}$ and $\mathcal{C} = \{\{1\}, \{2\}, \{1, 2\}\}$, which are both balanced. Take $\beta \in \Lambda^+$ such that $\beta_{\{1\}} = \beta_{\{2\}} = 1$ and $\beta_S = 0$ for $S \in \mathcal{N}\backslash\{\{1\},\{2\}\}$, and take $\gamma \in \Lambda^+$ such that $\gamma_{\{1,2\}} = 1$ and $\gamma_S = 0$ for $S \in \mathcal{N}\backslash\{\{1,2\}\}$. We have $\Lambda^+(\mathcal{B}) = \{\beta\}$ while $\Lambda^+(\mathcal{C}) = \{a\beta + (1-a)\gamma \mid\ a \in (0,1)\}$. $\triangleleft$

Now, for a vector $\beta \in \mathbb{R}^{\mathcal{N}}$, we define the set

$$V(\beta) = \{v \in \mathrm{TU}^N \mid \sum_{S\in\mathcal{N}} \beta_S v(S) \leq v(N)\}$$

of transferable utility games with player set N for which the weighted sum of the values of the coalitions with respect to β is less than or equal to the worth of the grand coalition. Also, we define $V^+(\mathcal{B}) = \cap_{\beta\in\Lambda^+(\mathcal{B})} V(\beta)$ and $V^+ = \cap_{\mathcal{B}\in\mathbb{B}^N} V^+(\mathcal{B})$. So, $V^+(\mathcal{B})$ is the set of games with player set N that satisfy the constraints imposed by all balanced weight vectors for collection $\mathcal{B}$, and V^+ is the set of games with player set N that satisfy the constraints imposed by all balanced weight vectors.

Consider some $\mathcal{B} \in \mathbb{B}^N$. Note that $v \in V(\beta)$ for some $\beta \in \Lambda^+(\mathcal{B})$ does not imply that $v \in V^+(\mathcal{B})$. This is illustrated by the following example.

Example 4.2.3 Consider the three person game $v \in \mathrm{TU}^N$ given by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v(S)$	2	0	0	8	8	4	8

We find that the balanced collection $\mathcal{B} = \{\{1\},\{1,2\},\{1,3\},\{2,3\}\}$ corresponds with more than one balanced weight vector, for instance $\beta,\gamma \in \Lambda^+(\mathcal{B})$ such that $\beta_{\{1\}} = \frac{1}{2}$, $\beta_{\{1,2\}} = \beta_{\{1,3\}} = \frac{1}{4}$, $\beta_{\{2,3\}} = \frac{3}{4}$, $\gamma_{\{1\}} = \frac{1}{4}$, $\gamma_{\{1,2\}} = \gamma_{\{1,3\}} = \frac{3}{8}$, $\gamma_{\{2,3\}} = \frac{5}{8}$ and $\beta_S = \gamma_S$ for every $S \in \mathcal{N}\backslash\mathcal{B}$. We have that

$$\sum_{S\in\mathcal{B}} \beta_S v(S) = \frac{1}{2}v(\{1\}) + \frac{1}{4}v(\{1,2\}) + \frac{1}{4}v(\{1,3\}) + \frac{3}{4}v(\{2,3\}) = 8 = v(N),$$

but

$$\sum_{S\in\mathcal{B}} \gamma_S v(S) = \frac{1}{4}v(\{1\}) + \frac{3}{8}v(\{1,2\}) + \frac{3}{8}v(\{1,3\}) + \frac{5}{8}v(\{2,3\}) = 9 > v(N).$$

So, $v \in V(\beta)$ but $v \notin V(\gamma)$. This implies that $v \notin V^+(\mathcal{B})$. $\triangleleft$

We call a game $v \in \mathrm{TU}^N$ balanced if $v \in V^+$.

Theorem 4.2.4 (Bondareva (1963), Shapley (1967)) Let $v \in \mathrm{TU}^N$. Then $C(v) \neq \emptyset$ if and only if $v \in V^+$.

It is well known that not all balanced collections are necessary to guarantee that a game is balanced. Minimal balanced collections suffice to characterize the class of games with a non-empty core.

Definition 4.2.5 A collection $\mathcal{B} \in \mathbb{B}^N$ is called *minimal balanced* if there does not exist a $\mathcal{C} \subsetneq \mathcal{B}$ such that $\mathcal{C} \in \mathbb{B}^N$. The class of minimal balanced collections on player set N is denoted by $\mathbb{B}^N_{\min}$.

Note that in Example 4.2.2, only the collection $\{\{1\},\{2\}\}$ is minimal balanced. We define $V^+_{\min} = \cap_{\mathcal{B} \in \mathbb{B}^N_{\min}} V^+(\mathcal{B})$ as the class of games that satisfy the constraints originating from minimal balanced collections.

Theorem 4.2.6 (Bondareva (1963), Shapley (1967)) Let $v \in \mathrm{TU}^N$. Then $C(v) \neq \emptyset$ if and only if $v \in V^+_{\min}$, i.e. $V^+ = V^+_{\min}$.

The following theorem shows an additional advantage of minimal balanced collections. Not only do we need just the minimal balanced collections to characterize the non-emptiness of the core, but also for every minimal balanced collection there exists only one balanced vector of weights. For the theorem, we provide the proof by Peleg and Sudhölter (2003) as we will use a similar technique later on to prove results on minimal exact balanced collections.

Theorem 4.2.7 (Bondareva (1963), Shapley (1967)) A collection $\mathcal{B} \in \mathbb{B}^N$ is minimal balanced if and only if $|\Lambda^+(\mathcal{B})| = 1$.

Proof: Let $\mathcal{B} \in \mathbb{B}^N$. Take $\beta \in \Lambda^+(\mathcal{B})$.
First we show that a balanced collection that is not minimal corresponds to more than one balanced weight vector. If $\mathcal{C} \subsetneq \mathcal{B}$ is a balanced collection with weights $\gamma \in \Lambda^+(\mathcal{C})$, then it is readily verified that $a\gamma + (1-a)\beta \in \Lambda^+(\mathcal{B})$ for $a \in [0,1)$, so the weight vector for $\mathcal{B}$ is not unique.
Second, we show that every collection with more than one balanced weight vector is not minimal. Assume that there exists another weight vector $\alpha \in \Lambda(\mathcal{B})$, $\alpha \neq \beta$. As there exists a coalition $S \in \mathcal{B}$ such that $\beta_S > \alpha_S$, we obtain that $a = \min\{\frac{\alpha_S}{\beta_S - \alpha_S} \mid$

$\beta_S > \alpha_S\}$ is well defined. Let $\gamma_S = (1 + a)\alpha_S - a\beta_S$ for all $S \in \mathcal{B}$. Then $\mathcal{C} = \{S \in \mathcal{B} \mid \gamma_S > 0\}$ is a proper subcollection of $\mathcal{B}$ with $\gamma \in \Lambda^+(\mathcal{C})$. So, $\mathcal{C} \in \mathbb{B}^N$ and $\mathcal{B}$ is not minimal. □

The following theorem states that we cannot characterize the set of balanced games by a subset of the minimal balanced collections.

Theorem 4.2.8 (Bondareva (1963), Shapley (1967)) Let $\mathcal{B} \in \mathbb{B}^N_{\min}$. Then there exists a game $v \in \mathrm{TU}^N$ such that $v \in V^+(\mathcal{C})$ for all collections $\mathcal{C} \in \mathbb{B}^N_{\min} \backslash \{\mathcal{B}\}$ and $v \notin V^+(\mathcal{B})$.

4.3 Exact balancedness

Games with a non-empty core can be characterized using balanced collections. A similar characterization exists for exact games. Exact games form a subclass of the class of games with a non-empty core.

Definition 4.3.1 A game $v \in \mathrm{TU}^N$ is *exact* if for every coalition $S \in \mathcal{N}$ there exists an $x \in C(v)$ such that $\sum_{i \in S} x_i = v(S)$.

Schmeidler (1972) and Csóka et al. (2011) provide characterizations of exact games, using concepts related to balancedness. Here we use the characterization called exact balancedness as defined by Csóka et al. (2011), except that in line with the definition of balancedness, we exclude the trivial collection $\{N\}$.

Definition 4.3.2 For a collection $\mathcal{E} \subseteq \mathcal{N}$, $\mathcal{E} \neq \{N\}$, a vector of weights $\lambda \in \mathbb{R}^{\mathcal{N}}$ is called *exact balanced* if there exists a $T \in \mathcal{E}$ such that $\lambda_S > 0$ for all $S \in \mathcal{E} \backslash \{T\}$, $\lambda_T \neq 0$, $\lambda_S = 0$ for all $S \notin \mathcal{E}$, and $\sum_{S \in \mathcal{E}} \lambda_S e^S = e^N$. We denote $\Lambda(\mathcal{E})$ for the set of all exact balanced vectors on $\mathcal{E}$. A collection $\mathcal{E} \subseteq \mathcal{N}$ is called *exact balanced* if $\Lambda(\mathcal{E}) \neq \emptyset$. Denote $\mathbb{E}^N$ for the set of all exact balanced collections on player set N, and $\Lambda = \cup_{\mathcal{E} \in \mathbb{E}^N} \Lambda(\mathcal{E})$.

In the remainder, we will typically use $\mathcal{E}$ and $\mathcal{D}$ to denote exact balanced collections, and use λ and δ to denote their respective weight vectors.

Note the discrepancy with the definition of balanced vectors. For exact balanced weight vectors, we allow for one negative weight. It is readily checked that $\Lambda^+(\mathcal{E}) \subseteq \Lambda(\mathcal{E})$ for every $\mathcal{E} \subseteq \mathcal{N}$, and therefore $\mathbb{B}^N \subseteq \mathbb{E}^N$. In contrast with Λ^+, Λ in general is not a convex set.

Example 4.3.3 Let $N = \{1,2,3\}$. Take $\lambda, \delta \in \mathbb{R}^{\mathcal{N}}$ such that $\lambda_{\{1,2\}} = \lambda_{\{1,3\}} = 1$, $\lambda_{\{1\}} = -1$ and $\delta_{\{1,2\}} = \delta_{\{2,3\}} = 1$, $\delta_{\{2\}} = -1$. Clearly, λ and δ are exact balanced weight vectors. However, the convex combination $\frac{1}{2}(\lambda + \delta)$ is not an exact balanced weight vector, as it has two negative components. This means that Λ is not a convex set. $\triangleleft$

Define, similar to the definitions of $V^+(\mathcal{B})$ and V^+, $V(\mathcal{E}) = \cap_{\lambda \in \Lambda(\mathcal{E})} V(\lambda)$ for all $\mathcal{E} \in \mathbb{E}^N$ and $V = \cap_{\mathcal{E} \in \mathbb{E}^N} V(\mathcal{E})$. Note that $V(\lambda)$ was already defined for all $\lambda \in \mathbb{R}^{\mathcal{N}}$ as the class of games that satisfy the constraint imposed by weight vector λ. So, $V(\mathcal{E})$ is the set of all games that satisfy the constraints imposed by the exact balanced weight vectors of the collection $\mathcal{E}$ and V is the class of exact balanced games.

Theorem 4.3.4 (Csóka et al. (2011)) A game $v \in \mathrm{TU}^N$ is exact if and only if $v \in V$.

So, just as balancedness is equivalent with non-emptiness of the core, we have that exact balancedness is equivalent with the existence of a core element for every coalition, where this coalition gets precisely its own value in the game. Similar to the definition of minimal balanced collections, we define minimal exact balanced collections.

Definition 4.3.5 A collection $\mathcal{E} \in \mathbb{E}^N$ is *minimal exact balanced* if there exists no $\mathcal{D} \subsetneq \mathcal{E}$ such that $\mathcal{D} \in \mathbb{E}^N$. We denote $\mathbb{E}^N_{\min}$ for the class of minimal exact balanced collections.

For two-player games, the class of minimal exact balanced collections coincides with the class of minimal balanced collections.

Example 4.3.6 Regarding exact balancedness, a similar reasoning as in Example 4.2.2 can be used to show that only $\{\{1\},\{2\}\}$ and $\{\{1\},\{2\},\{1,2\}\}$ are exact balanced for two-person games. So, $\mathbb{E}^N = \mathbb{B}^N$ and $\mathbb{E}^N_{\min} = \mathbb{B}^N_{\min}$. We have

$\Lambda(\{\{1\},\{2\}\}) = \Lambda^+(\{\{1\},\{2\}\})$. However, $\Lambda(\{\{1\},\{2\},\{1,2\}\}) = \{\lambda^a\}_{a\in\mathbb{R}_{++}\setminus\{1\}}$ and $\Lambda^+(\{\{1\},\{2\},\{1,2\}\}) = \{\lambda^a\}_{a\in(0,1)}$ where $\lambda^a_{\{1\}} = \lambda^a_{\{2\}} = a$ and $\lambda^a_{\{1,2\}} = 1-a$. Take $\beta \in \Lambda^+(\{\{1\},\{2\}\})$. Since for every $a > 1$, $v \in V(\lambda^a)$ is implied by $v \in V(\beta)$, we have $V^+ = V$. This is not surprising, since for two-player games, whenever the core is non-empty there exists a core element where player 1 gets $v(\{1\})$ and there exists a core element where player 2 gets $v(\{2\})$. So, the concepts of balancedness and exactness are equivalent for two player games. For games with three or more players, $\mathbb{B}^N \subsetneq \mathbb{E}^N$. For $N = \{1,2,3\}$, $\mathbb{E}^N_{\min}$ and $\mathbb{B}^N_{\min}$ are given in Table 4.3.1.

$\mathbb{B}^N_{\min}$	$\mathbb{E}^N_{\min}$
$\{1\},\{2\},\{3\}$	$\{1\},\{2\},\{3\}$
$\{1,2\},\{3\}$	$\{1,2\},\{3\}$
$\{1,3\},\{2\}$	$\{1,3\},\{2\}$
$\{2,3\},\{1\}$	$\{2,3\},\{1\}$
$\{1,2\},\{1,3\},\{2,3\}$	$\{1,2\},\{1,3\},\{2,3\}$
	$\{1\},\{1,2\},\{1,3\}$
	$\{2\},\{1,2\},\{2,3\}$
	$\{3\},\{1,3\},\{2,3\}$
	$\{1\},\{2\},\{1,2\},N$
	$\{1\},\{3\},\{1,3\},N$
	$\{2\},\{3\},\{2,3\},N$

Table 4.3.1: Minimal balanced and minimal exact balanced collections for $N = \{1,2,3\}$. $\triangleleft$

If the size of the player set increases, the number of collections in the different classes grows considerably. Table 4.3.2 shows the number of collections in all classes for up to four players. These collections are generated using the results of Section 4.4. Appendix 4.A contains all minimal exact balanced collections and the corresponding weight vectors for three- and four-player games. Here, the minimal exact balanced collections are partitioned in three classes that will be introduced in Section 4.4: minimal balanced collections, minimal negative balanced collections and minimal subbalanced collections. As is stated in Theorem 4.2.7, the class of minimal balanced collections coincides with the set of balanced collections with a unique weight vector. For minimal exact balanced collections, a somewhat weaker statement holds: the class of minimal exact balanced collections not containing the grand coalition coincides with the set of exact balanced collections with a unique weight vector.

$\|N\|$	3	4
$\|\mathbb{B}^N\|$	42	18878
$\|\mathbb{B}^N_{\text{min}}\|$	5	41
$\|\mathbb{E}^N\|$	63	27014
$\|\mathbb{E}^N_{\text{min}}\|$	11	165

Table 4.3.2: Number of collections in the different classes.

Theorem 4.3.7 Let $\mathcal{E} \in \mathbb{E}^N$. Then $|\Lambda(\mathcal{E})| = 1$ if and only if both $\mathcal{E} \in \mathbb{E}^N_{\text{min}}$ and $N \notin \mathcal{E}$.

Proof: To prove the 'only if' part of the theorem, let $\mathcal{E} \subseteq \mathcal{N}$ be such that $\Lambda(\mathcal{E}) = \{\lambda\}$ for some $\lambda \in \mathbb{R}^{\mathcal{N}}$. First suppose $\mathcal{E} \notin \mathbb{E}^N_{\text{min}}$. We show that we can construct a second weight vector in $\Lambda(\mathcal{E})$. As $\mathcal{E} \notin \mathbb{E}^N_{\text{min}}$, there exists an exact balanced subcollection $\mathcal{D} \subsetneq \mathcal{E}$. Take $\mu \in \Lambda(\mathcal{D})$ and define the function $f : [0,1] \to \mathbb{R}^{\mathcal{N}}$ by $f(b) = (1-b)\lambda + b\mu$. As f is continuous, there exists an $\varepsilon > 0$ such that the sign of $f_S(\varepsilon)$ coincides with the sign of λ_S for all $S \in \mathcal{E}$. Since $\sum_{S\in\mathcal{E}} f_S(\varepsilon)e^S = \sum_{S\in\mathcal{E}}(1-\varepsilon)\lambda_S e^S + \sum_{S\in\mathcal{D}} \varepsilon\mu_S e^S = e^N$, we obtain that $f(\varepsilon) \in \Lambda(\mathcal{E})$ while $f(\varepsilon) \neq \lambda$, a contradiction.

Secondly, suppose $\mathcal{E} \in \mathbb{E}^N_{\text{min}}$ and $N \in \mathcal{E}$. It is readily checked that $\lambda_N \leq 1$, and if $\lambda_N < 1$ we obtain that the collection $\mathcal{A} = \mathcal{E}\backslash\{N\}$ is exact balanced with weight vector $\mu_S = \frac{\lambda_S}{1-\lambda_N}$ for every $S \in \mathcal{A}$ which contradicts $\mathcal{E} \in \mathbb{E}^N_{\text{min}}$. Hence, $\lambda_N = 1$. As $\sum_{S\in\mathcal{E}\backslash\{N\}} \lambda_S e^S = 0$, we have that $\sum_{S\in\mathcal{E}\backslash\{N\}} 2\lambda_S e^S = 0$. Define the weight vector μ by $\mu_S = 2\lambda_S$ for all $S \in \mathcal{E}\backslash\{N\}$, $\mu_N = 1$ and $\mu_S = 0$ otherwise. It is readily checked that $\mu \in \Lambda(\mathcal{E})$ with $\mu \neq \lambda$, a contradiction.

We prove the 'if' part of the theorem by showing that we can construct an exact balanced subcollection of $\mathcal{E}$ if the weight vector is not unique. Take $\mathcal{E} \in \mathbb{E}^N_{\text{min}}$ with $N \notin \mathcal{E}$. Suppose that there exist two weight vectors $\lambda, \mu \in \Lambda(\mathcal{E})$ such that $\lambda \neq \mu$.

If both $\lambda \in \Lambda^+(\mathcal{E})$ and $\mu \in \Lambda^+(\mathcal{E})$, we have by Theorem 4.2.7 that $\mathcal{E} \notin \mathbb{B}^N_{\text{min}}$. Hence, there exists an exact balanced subcollection of $\mathcal{E}$ in this case.

Next assume $\mathcal{E} \in \mathbb{B}^N_{\text{min}}$, $\lambda \in \Lambda^+(\mathcal{E})$ and $\mu \notin \Lambda^+(\mathcal{E})$. Let $U \in \mathcal{E}$ be such that $\mu_U < 0$, and take $a = \min\{\frac{\lambda_S}{\mu_S} \mid S \in \mathcal{E}\backslash\{U\}\}$ and $\beta = \frac{1}{1-a}(\lambda - a\mu)$. Note that $0 < a < 1$ since $\lambda_S > 0$ and $\mu_S > 0$ for all $S \in \mathcal{E}\backslash\{U\}$, and $a \geq 1$ would imply that

$$\begin{aligned} e^N = \sum_{S\in\mathcal{E}} \mu_S e^S &= \textstyle\sum_{S\in\mathcal{E}\backslash\{U\}} \mu_S e^S + \mu_U e^U < \sum_{S\in\mathcal{E}\backslash\{U\}} \lambda_S e^S + \lambda_U e^U \\ &= \textstyle\sum_{S\in\mathcal{E}} \lambda_S e^S \leq e^N, \end{aligned}$$

where the strict inequality uses that $\mu_U < 0 < \lambda_U$. Note that $\beta_S = \frac{1}{1-a}(\lambda_S - a\mu_S) \geq 0$ for all $S \in \mathcal{E}$, with equality for at least one coalition. If we take $\mathcal{B} = \{S \in \mathcal{E} \mid \beta_S > 0\}$, then $\mathcal{B}$ is a proper subset of $\mathcal{E}$ and $\sum_{S\in\mathcal{B}} \beta_S e^S = \sum_{S\in\mathcal{E}} \beta_S e^S = \sum_{S\in\mathcal{E}} \left(\frac{\lambda_S}{1-a} e^S - \frac{a\mu_S}{1-a} e^S\right) = e^N$, so $\mathcal{B} \in \mathbb{B}^N$ which contradicts $\mathcal{E} \in \mathbb{B}^N_{\min}$.
Finally, let $\lambda \notin \Lambda^+(\mathcal{E})$ and $\mu \notin \Lambda^+(\mathcal{E})$. This means that there exist coalitions $T \in \mathcal{E}$ and $U \in \mathcal{E}$ such that $\lambda_T < 0$ and $\mu_U < 0$.
Assume $T = U$. Take $a = \min\{\frac{\lambda_S}{\mu_S} \mid S \in \mathcal{E}\}$. Note that $a > 0$ since for $S \in \mathcal{E}$ either both $\lambda_S > 0$ and $\mu_S > 0$ or both $\lambda_S < 0$ and $\mu_S < 0$. It holds that $a < 1$, as $a \geq 1$ would imply that either $\lambda = \mu$ or $\sum_{S\ni i} \lambda_S > \sum_{S\ni i} \mu_S = 1$ for $i \in N\backslash T$, a non-empty set since by assumption $N \notin \mathcal{E}$. We construct $\delta = \frac{1}{1-a}\lambda - \frac{a}{1-a}\mu$ and $\mathcal{D} = \{S \in \mathcal{E} \mid \delta_S \neq 0\}$. It is readily verified that $\delta_S \geq 0$ for all $S \in \mathcal{D}\backslash\{T\}$ and $\sum_{S\in\mathcal{D}} \delta_S e^S = e^N$. This shows that $\mathcal{D} \in \mathbb{E}^N$ and by construction $\mathcal{D} \subsetneq \mathcal{E}$, which contradicts $\mathcal{E} \in \mathbb{E}^N_{\min}$.
Now assume $T \neq U$. Take $a = \frac{\mu_T}{\mu_T - \lambda_T}$. It is readily checked that $0 < a < 1$. Take $\delta = a\lambda + (1-a)\mu$ and $\mathcal{D} = \{S \in \mathcal{E} \mid \delta_S \neq 0\}$. We have $\delta_S > 0$ for every $S \in \mathcal{D}\backslash\{T, U\}$ and $\delta_T = 0$. Since $\sum_{S\in\mathcal{D}} \delta_S e^S = \sum_{S\in\mathcal{E}} a\lambda_S e^S + \sum_{S\in\mathcal{E}} (1-a)\mu_S e^S = e^N$ this shows that $\mathcal{D} \in \mathbb{E}^N$ which contradicts $\mathcal{E} \in \mathbb{E}^N_{\min}$. □

As Example 4.3.8 shows, there exist minimal exact balanced collections with more than one exact balanced weight vector. By Theorem 4.3.7 such a collection must contain the set N.

Example 4.3.8 Take $N = \{1, 2, 3\}$. The collection $\mathcal{E} = \{\{1\}, \{2\}, \{1, 2\}, N\}$ is minimal exact balanced, but there exists more than one weight vector: define λ by $\lambda_{\{1\}} = \lambda_{\{2\}} = 1$, $\lambda_{\{1,2\}} = -1$ and $\lambda_N = 1$ and μ by $\mu_{\{1\}} = \mu_{\{2\}} = 2$, $\mu_{\{1,2\}} = -2$ and $\mu_N = 1$. It is readily checked that $\lambda \in \Lambda(\mathcal{E})$ and $\mu \in \Lambda(\mathcal{E})$. ◁

If a minimal exact balanced collection does contain the grand coalition, then there exists more than one exact balanced weight vector, but these weight vectors are related in a special way and induce the same constraint on the game. Furthermore, if for an exact balanced collection all weight vectors induce the same constraint on the game, then the collection is minimal exact balanced.

Theorem 4.3.9 Let $\mathcal{E} \in \mathbb{E}^N$. Then both $\mathcal{E} \in \mathbb{E}^N_{\min}$ and $N \in \mathcal{E}$ if and only if for every $\lambda \in \Lambda(\mathcal{E})$ and $\mu \in \Lambda(\mathcal{E})$ there exists a scalar $a > 0$ such that

$$\begin{aligned}\mu_S &= a\lambda_S \text{ for all } S \in \mathcal{E}\backslash\{N\},\\ \mu_N &= \lambda_N = 1.\end{aligned}$$

Proof: For the 'only if' part of the proof, assume $\mathcal{E} \in \mathbb{E}^N_{\min}$ and $N \in \mathcal{E}$. Let $\lambda \in \Lambda(\mathcal{E})$. It is readily checked that $\lambda_N \leq 1$, and if $\lambda_N < 1$ we obtain that the collection $\mathcal{C} = \mathcal{E}\backslash\{N\}$ is exact balanced with weight vector $\gamma_S = \frac{\lambda_S}{1-\lambda_N}$ for every $S \in \mathcal{C}$ and $\gamma_S = 0$ otherwise. Hence, $\lambda_N = 1$.
Take $T \in \mathcal{E}$ such that $\lambda_T < 0$. Such a $T \in \mathcal{E}$ exists, as $N \in \mathcal{E}$ and therefore $\mathcal{E} \notin \mathbb{B}^N_{\min}$. As $\sum_{S\in\mathcal{E}\backslash\{N\}} \lambda_S e^S = 0$ and $\lambda_S > 0$ for all $S \in \mathcal{E}\backslash\{T\}$, we obtain that $S \subsetneq T$ for all $S \in \mathcal{E}\backslash\{T, N\}$. This implies that the location of the negative weight is unique, $\mu_T < 0$ for every $\mu \in \Lambda(\mathcal{E})$. Rewriting $\sum_{S\in\mathcal{E}\backslash\{N\}} \lambda_S e^S = 0$ yields $\sum_{S\in\mathcal{E}\backslash\{N,T\}} -\frac{\lambda_S}{\lambda_T} e^S = e^T$, and therefore $\mathcal{E}\backslash\{N, T\} \in \mathbb{B}^T$. If there exists a minimal balanced collection $\mathcal{B} \in \mathbb{B}^T_{\min}$ such that $\mathcal{B} \subsetneq \mathcal{E}\backslash\{T, N\}$, it is readily checked that $\mathcal{B} \cup \{T, N\}$ is an exact balanced collection, which contradicts our assumption of $\mathcal{E} \in \mathbb{E}^N_{\min}$. Hence, $\mathcal{E}\backslash\{N, T\} \in \mathbb{B}^T_{\min}$. Since $\mathcal{E}\backslash\{N, T\} \in \mathbb{B}^T_{\min}$, by Theorem 4.2.7 there is a unique balanced vector of weights β of $\mathcal{E}\backslash\{N, T\}$. Note that

$$\begin{aligned}e^N &= e^N + \lambda_T e^T + \sum_{S\in\mathcal{E}\backslash\{N,T\}} \lambda_S e^S\\ &= e^N + \lambda_T \sum_{S\in\mathcal{E}\backslash\{N,T\}} \beta_S e^S + \sum_{S\in\mathcal{E}\backslash\{N,T\}} \lambda_S e^S\\ &= e^N + \sum_{S\in\mathcal{E}\backslash\{N,T\}} (\lambda_T\beta_S + \lambda_S) e^S.\end{aligned}$$

This implies that $\sum_{S\in\mathcal{E}\backslash\{N,T\}} (\lambda_T\beta_S + \lambda_S)e^S = 0$. If $\lambda_T\beta_S \neq \lambda_S$ for some $S \in \mathcal{E}\backslash\{N, T\}$ we have $\gamma \in \mathbb{B}^T_{\min}$ where we define for small $\varepsilon > 0$, $\gamma_S = \beta_S + \varepsilon(\lambda_T\beta_S + \lambda_S)$ for every $S \in \mathcal{E}\backslash\{N, T\}$ and $\gamma_S = 0$ otherwise. So, $\lambda_T\beta_S + \lambda_S = 0$ and therefore $\lambda_S = -\lambda_T\beta_S$ for every $S \in \mathcal{E}\backslash\{N, T\}$. Now take $\mu \in \Lambda(\mathcal{E})$ and take $a = \frac{\mu_T}{\lambda_T}$. Since $\mu_T < 0$ and $\lambda_T < 0$, $a > 0$. We have $\mu_T = a\lambda_T$ by definition, and $\mu_S = -\mu_T\beta_S = -a\lambda_T\beta_S = \lambda_S$ for every $S \in \mathcal{E}\backslash\{N, T\}$.

For the 'if' part of the proof, assume that for every $\lambda \in \Lambda(\mathcal{E})$ and $\mu \in \Lambda(\mathcal{E})$, $\lambda_N = \mu_N = 1$ and there exists an $a > 0$ such that $\mu_S = a\lambda_S$ for every $S \in \mathcal{E}\backslash\{N\}$. Clearly we have $N \in \mathcal{E}$. Suppose $\mathcal{E} \notin \mathbb{E}^N_{\min}$. As $\mathcal{E}$ is not minimal, there exists a $\mathcal{D} \subsetneq \mathcal{E}$ such that $\mathcal{D} \in \mathbb{E}^N_{\min}$. Let $\lambda \in \Lambda(\mathcal{E})$ and $\delta \in \Lambda(\mathcal{D})$. Define $\mu = (1-b)\lambda + b\delta$, where $b > 0$ is sufficiently small, such that the sign of δ_S equals the sign of μ_S for every $S \in \mathcal{E}$. Clearly, $\mu \in \Lambda(\mathcal{E})$. Take $T \in \mathcal{E}\backslash\mathcal{D}$ and $U \in \mathcal{D}$, $U \neq N$. Such a

U exists, as $\{N\}$ is not a minimal exact balanced collection by definition. Since $\mu_T = (1-b)\lambda_T$ and $\mu_U = (1-b)\lambda_U + b\delta_U \neq (1-b)\lambda_U$, there does not exist a scalar $a > 0$ such that $\mu_T = a\lambda_T$ and $\mu_U = a\lambda_U$, which is a contradiction. Hence, $\mathcal{E} \in \mathbb{E}^N_{\min}$. As we showed before that $N \in \mathcal{E}$, this completes the proof. $\square$

For minimal balanced collections, the corresponding weight vector is unique. Similarly, we have shown that for minimal exact balanced collections either the corresponding weight vector is unique or all corresponding weight vectors induce the same constraint on the game. This way, for every minimal exact balanced collection we can use one *standardized weight vector*. In the remainder, for every minimal balanced collection $\mathcal{B}$ we denote $\beta^{\mathcal{B}}$ for the unique balanced weight vector. More general, for every $\mathcal{E} \in \mathbb{E}^N_{\min}$ with $N \notin \mathcal{E}$, we denote $\lambda^{\mathcal{E}}$ for the unique exact balanced weight vector. For $\mathcal{E} \in \mathbb{E}^N_{\min}$ with $N \in \mathcal{E}$, $\lambda^{\mathcal{E}}$ denotes the standardized exact balanced weight vector such that $\min\{\lambda^{\mathcal{E}}_S \mid S \in \mathcal{E}\} = -1$. Notice that for notational convenience, for $\mathcal{B} \in \mathbb{B}^N_{\min}$ the standardized weight vector is both denoted by $\beta^{\mathcal{B}}$ and $\lambda^{\mathcal{B}}$.

4.4 Partitioning the class of minimal exact balanced collections

In this section we study the structure of the class of minimal exact balanced collections. It turns out that this set can be decomposed in three parts, all related to balanced collections. The first part consists of all minimal balanced collections.

Theorem 4.4.1 $\mathbb{B}^N_{\min} \subseteq \mathbb{E}^N_{\min}$.

Proof: Let $\mathcal{B} \in \mathbb{B}^N_{\min}$. It is clear that every minimal balanced collection is also exact balanced. It remains to show that it is also minimal exact balanced. Assume there exists an exact balanced collection $\mathcal{E} \subsetneq \mathcal{B}$ and take $\lambda \in \Lambda(\mathcal{E})$. We will show that this results in a contradiction with $\mathcal{B} \in \mathbb{B}^N_{\min}$.
Since $\mathcal{B} \in \mathbb{B}^N_{\min}$ we know that there exists a $T \in \mathcal{E}$ such that $\lambda_T < 0$ as $\mathcal{B}$ does not have a proper subset that is balanced. Take $a = \min\{\frac{\beta^{\mathcal{B}}_S}{\lambda_S} \mid S \in \mathcal{E}\backslash\{T\}\}$ and $\gamma = \frac{1}{1-a}(\beta^{\mathcal{B}} - a\lambda)$. Note that $0 < a < 1$ since $\beta^{\mathcal{B}}_S > 0$ and $\lambda_S > 0$ for all $S \in \mathcal{E}\backslash\{T\}$, and $a \geq 1$ would imply that

$$\begin{aligned}\sum_{S\in\mathcal{E}}\lambda_S e^S &= \sum_{S\in\mathcal{E}\backslash\{T\}}\lambda_S e^S+\lambda_T e^T,\\ &< \sum_{S\in\mathcal{E}\backslash\{T\}}\beta_S^{\mathcal{B}} e^S+\beta_T^{\mathcal{B}} e^T,\\ &= \sum_{S\in\mathcal{E}}\beta_S^{\mathcal{B}} e^S,\\ &\leq e^N.\end{aligned}$$

Now $\gamma_S \geq 0$ for all $S \in \mathcal{B}$, with equality for at least one coalition. Define the collection $\mathcal{C} = \{S \in \mathcal{E} \mid \gamma_S > 0\}$. Then $\mathcal{C}$ is a proper subset of $\mathcal{B}$ and

$$\sum_{S\in\mathcal{C}}\gamma_S e^S=\sum_{S\in\mathcal{B}}\gamma_S e^S=\sum_{S\in\mathcal{B}}\frac{\beta_S^{\mathcal{B}}}{1-a}e^S-\sum_{S\in\mathcal{E}}\frac{a\lambda_S}{1-a}e^S=\frac{1}{1-a}e^N-\frac{a}{1-a}e^N=e^N,$$

so $\mathcal{C}\in\mathbb{B}^N$, contradicting $\mathcal{B}\in\mathbb{B}^N_{\min}$. □

The second part of the partition of $\mathbb{E}^N_{\min}$ consists of so-called *negative balanced* collections. The set of all negative balanced collections is denoted by $\overline{\mathbb{B}}^N_{\min}$. The negative balanced collections can be obtained, by replacing one coalition in a minimal balanced collection by its complement. However, this is only allowed for the coalitions with weight strictly smaller than 1. We have

$$\overline{\mathbb{B}}^N_{\min}=\{(\mathcal{B}\backslash\{S\})\cup\{N\backslash S\}\mid \mathcal{B}\in\mathbb{B}^N_{\min}, S\in\mathcal{B}:\beta_S^{\mathcal{B}}<1\}.$$

Example 4.4.2 Let $N = \{1,2,3,4\}$, and consider the minimal balanced collection $\mathcal{B} = \{\{1,2\},\{1,3\},\{2,3\},\{4\}\}$. For the weight vector $\beta^{\mathcal{B}}$ it holds that $\beta^{\mathcal{B}}_{\{1,2\}} = \frac{1}{2}$. This means that $\mathcal{E} = (\mathcal{B}\backslash\{\{1,2\}\})\cup(\{\{3,4\}\}) = \{\{3,4\},\{1,3\},\{2,3\},\{4\}\}\in\overline{\mathbb{B}}^N_{\min}$. It is readily checked that $\mathcal{E}\in\mathbb{E}^N$, since $e^{\{1,3\}}+e^{\{2,3\}}+2e^{\{4\}}-e^{\{3,4\}} = e^N$. As $\beta^{\mathcal{B}}_{\{4\}} = 1$, we cannot replace the coalition $\{4\}$ by its complement to obtain an element of $\overline{\mathbb{B}}^N_{\min}$. ◁

The exact balanced weight vector of a negative balanced collection can be derived from the balanced weight vector of the corresponding balanced collection.

Theorem 4.4.3 Let $\mathcal{E}\in\overline{\mathbb{B}}^N_{\min}$. Let $\mathcal{B}\in\mathbb{B}^N_{\min}$ and $U\in\mathcal{B}$ be such that $\mathcal{E} = (\mathcal{B}\backslash\{U\})\cup\{N\backslash U\}$ and $\beta^{\mathcal{B}}_U<1$. Let $\lambda_S=\frac{\beta^{\mathcal{B}}_S}{1-\beta^{\mathcal{B}}_U}$ for all $S\in\mathcal{B}\backslash\{U\}$, $\lambda_{N\backslash U}=-\frac{\beta^{\mathcal{B}}_U}{1-\beta^{\mathcal{B}}_U}$ and $\lambda_S=0$ for $S\in\mathcal{N}\backslash\mathcal{E}$. Then $\lambda\in\Lambda(\mathcal{E})$.

Proof: As $\mathcal{B} \in \mathbb{B}^N_{\min}$ and $\beta^{\mathcal{B}}_U < 1$, we know $N \backslash U \notin \mathcal{B}$. As $0 < \beta^{\mathcal{B}}_U < 1$, we obtain that $\lambda_S = \frac{\beta^{\mathcal{B}}_S}{1-\beta^{\mathcal{B}}_U} > 0$ for all $S \in \mathcal{E} \backslash \{U\}$ and $\lambda_{N \backslash U} = -\frac{\beta^{\mathcal{B}}_U}{1-\beta^{\mathcal{B}}_U} < 0$. For $i \in U$,

$$\sum_{\substack{S\in\mathcal{E},\\ S\ni i}} \lambda_S = \sum_{\substack{S\in(\mathcal{B}\backslash\{U\})\cup\{N\backslash U\},\\ S\ni i}} \lambda_S = \sum_{\substack{S\in(\mathcal{B}\backslash\{U\}),\\ S\ni i}} \frac{\beta^{\mathcal{B}}_S}{1-\beta^{\mathcal{B}}_U} = \frac{1}{1-\beta^{\mathcal{B}}_U} \sum_{\substack{S\in\mathcal{B}\backslash\{U\},\\ S\ni i}} \beta^{\mathcal{B}}_S = 1,$$

and for $i \in N \backslash U$ it holds that

$$\sum_{\substack{S\in\mathcal{E},\\ S\ni i}} \lambda_S = \sum_{\substack{S\in(\mathcal{B}\backslash\{U\})\cup\{N\backslash U\},\\ S\ni i}} \lambda_S = \sum_{\substack{S\in\mathcal{B},\\ S\ni i}} \frac{\beta^{\mathcal{B}}_S}{1-\beta^{\mathcal{B}}_U} - \frac{\beta^{\mathcal{B}}_U}{1-\beta^{\mathcal{B}}_U} = \frac{1}{1-\beta^{\mathcal{B}}_U} \left(\sum_{\substack{S\in\mathcal{B},\\ S\ni i}} \beta^{\mathcal{B}}_S - \beta^{\mathcal{B}}_U \right) = 1.$$

So, indeed $\lambda \in \Lambda(\mathcal{E})$. $\square$

By definition of $\overline{\mathbb{B}}^N_{\min}$ and the observation that $N \notin \mathcal{B}$ for every $\mathcal{B} \in \mathbb{B}^N_{\min}$, we have $N \notin \mathcal{E}$ for every $\mathcal{E} \in \overline{\mathbb{B}}^N_{\min}$. Hence, for this second part of the partition we can focus on collections without the grand coalition. Consider such a collection which is not minimal balanced. Then it is minimal exact balanced if and only if it is negative balanced.

Theorem 4.4.4

(i) $\overline{\mathbb{B}}^N_{\min} \subseteq \mathbb{E}^N_{\min} \backslash \mathbb{B}^N_{\min}$.

(ii) Let $\mathcal{E} \in \mathbb{E}^N_{\min} \backslash \mathbb{B}^N_{\min}$ and $N \notin \mathcal{E}$. Then $\mathcal{E} \in \overline{\mathbb{B}}^N_{\min}$.

Proof:
(i) Let $\mathcal{E} \in \overline{\mathbb{B}}^N_{\min}$. Let $\mathcal{B} \in \mathbb{B}^N_{\min}$ and $U \in \mathcal{B}$ be such that $\mathcal{E} = (\mathcal{B} \backslash \{U\}) \cup \{N \backslash U\}$ and $\beta^{\mathcal{B}}_U < 1$. Let $\lambda_S = \frac{\beta^{\mathcal{B}}_S}{1-\beta^{\mathcal{B}}_U}$ for all $S \in \mathcal{B} \backslash \{U\}$, $\lambda_{N\backslash U} = -\frac{\beta^{\mathcal{B}}_U}{1-\beta^{\mathcal{B}}_U}$ and $\lambda_S = 0$ for $S \in \mathcal{N} \backslash \mathcal{E}$. From Theorem 4.4.3, it follows that $\lambda \in \Lambda(\mathcal{E})$ and therefore $\mathcal{E} \in \mathbb{E}^N$.
We prove that $\mathcal{E} \in \mathbb{E}^N_{\min}$. Assume on the contrary that there exists a subset $\mathcal{D} \subsetneq \mathcal{E}$, with $\mathcal{D} \in \mathbb{E}^N_{\min}$. By minimality of $\mathcal{B}$, it must hold that $N \backslash U \in \mathcal{D}$ as otherwise $\mathcal{D} \subsetneq \mathcal{B}$ which would contradict Theorem 4.4.1.
We distinguish two cases:

(1) Assume $\lambda^{\mathcal{D}}_S > 0$ for all $S \in \mathcal{D} \backslash \{N \backslash U\}$. We know $\lambda^{\mathcal{D}}_{N\backslash U} < 1$, since $\lambda^{\mathcal{D}}_{N\backslash U} = 1$ would mean that $\mathcal{D} \backslash \{N \backslash U\}$ is a balanced collection on U which contradicts minimality of $\mathcal{B}$ as we can omit U from $\mathcal{B}$. Given that $\lambda^{\mathcal{D}}_{N\backslash U} < 1$, we can

reverse the procedure for constructing $\mathcal{E}$: take $\mathcal{A} = (\mathcal{D}\backslash\{N\backslash U\}) \cup (\{U\})$ and take $\alpha_S = \frac{\lambda_S^{\mathcal{D}}}{1-\lambda_{N\backslash U}^{\mathcal{D}}}$ for all $S \in \mathcal{D}\backslash\{N\backslash U\}$ and $\alpha_U = -\frac{\lambda_{N\backslash U}^{\mathcal{D}}}{1-\lambda_{N\backslash U}^{\mathcal{D}}}$. We obtain $\alpha_S > 0$ for all $S \in \mathcal{A}\backslash\{U\}$ and $\alpha_U \neq 0$. Furthermore, for $i \in U$:

$$\sum_{S\in\mathcal{A},S\ni i} \alpha_S = -\frac{\lambda_{N\backslash\{U\}}^{\mathcal{D}}}{1-\lambda_{N\backslash U}^{\mathcal{D}}} + \sum_{S\in\mathcal{D}\backslash\{N\backslash U\},S\ni i} \frac{\lambda_S^{\mathcal{D}}}{1-\lambda_{N\backslash U}^{\mathcal{D}}} = 1,$$

and for $i \in N\backslash U$ it holds that

$$\sum_{S\in\mathcal{A},S\ni i} \alpha_S = \sum_{S\in\mathcal{D}\backslash\{N\backslash U\},S\ni i} \frac{\lambda_S^{\mathcal{D}}}{1-\lambda_{N\backslash U}^{\mathcal{D}}} = 1,$$

So, $\alpha \in \Lambda(\mathcal{A})$ and therefore $\mathcal{A} \in \mathbb{E}^N$. As $\mathcal{A} \subsetneq \mathcal{B}$ this contradicts our assumption of $\mathcal{B} \in \mathbb{E}^N_{\min}$.

(2) Assume $\lambda_T^{\mathcal{D}} < 0$ for some $T \in \mathcal{D}\backslash\{N\backslash U\}$, which means that $\mathcal{D} \in \mathbb{E}^N_{\min}\backslash\mathbb{B}^N_{\min}$ and $\lambda_{N\backslash U}^{\mathcal{D}} > 0$. Take $c = -\frac{\lambda_{N\backslash U}^{\mathcal{D}}}{\lambda_{N\backslash U}^{\mathcal{E}}}$, and take $T \in \mathcal{D}$ such that $\lambda_T^{\mathcal{D}} < 0$. We construct the weight vector γ with $\gamma_S = \frac{c}{1+c}\lambda_S^{\mathcal{E}} + \frac{1}{1+c}\lambda_S^{\mathcal{D}}$ for all $S \in \mathcal{E}$ and $\gamma_S = 0$ if $S \in \mathcal{N}\backslash\mathcal{E}$. Furthermore, take $\mathcal{C} = \{S \in \mathcal{E} \mid \gamma_S \neq 0\}$. By definition of γ, we obtain $\gamma_{N\backslash U} = 0$ and $\gamma_S > 0$ for all $S \in \mathcal{C}\backslash\{T\}$. So, $\mathcal{C} \subsetneq \mathcal{B}$ and $\gamma \in \Lambda(\mathcal{C})$. Therefore $\mathcal{B} \notin \mathbb{E}^N_{\min}$, a contradiction with Theorem 4.4.1.

So, we have $\mathcal{E} \in \mathbb{E}^N_{\min}$. From Theorem 4.3.7 it follows that $\Lambda(\mathcal{E}) = \{\lambda\}$. Since $\lambda_{N\backslash U}$, $\mathcal{E} \in \mathbb{E}^N_{\min}\backslash\mathbb{B}^N_{\min}$.

(ii) Let $\mathcal{E} \in \mathbb{E}^N_{\min}\backslash\mathbb{B}^N_{\min}$ and $N \notin \mathcal{E}$. We will identify $\mathcal{B} \in \mathbb{B}^N_{\min}$ and the coalition $T \in \mathcal{B}$ such that $\beta_T^{\mathcal{B}} < 1$ and $\mathcal{E} = (\mathcal{B}\backslash\{T\}) \cup \{N\backslash T\}$. Take $T \in \mathcal{E}$ such that $\lambda_T^{\mathcal{E}} < 0$. Take $\mathcal{B} = (\mathcal{E}\backslash\{T\}) \cup \{N\backslash T\}$ and define $\beta_S = \frac{\lambda_S^{\mathcal{E}}}{1-\lambda_T^{\mathcal{E}}}$ for all $S \in \mathcal{E}\backslash\{T\}$ and $\beta_{N\backslash T} = -\frac{\lambda_T^{\mathcal{E}}}{1-\lambda_T^{\mathcal{E}}}$. We obtain $\beta_S > 0$ for all $S \in \mathcal{B}$. Furthermore, for $i \in N\backslash T$:

$$\sum_{S\in\mathcal{B},S\ni i} \beta_S = -\frac{\lambda_T^{\mathcal{E}}}{1-\lambda_T^{\mathcal{E}}} - \sum_{S\in\mathcal{E},S\ni i} \frac{\lambda_S^{\mathcal{E}}}{1-\lambda_T^{\mathcal{E}}} = 1,$$

and for $i \in T$ it holds that

$$\sum_{S \in \mathcal{B}, S \ni i} \beta_S = \sum_{S \in \mathcal{E} \backslash \{T\}, S \ni i} \frac{\lambda_S^{\mathcal{E}}}{1 - \lambda_T^{\mathcal{E}}} = 1,$$

So, $\mathcal{B} \in \mathbb{B}^N$. It remains to show that $\mathcal{B}$ is minimal. Here we need the condition that $N \notin \mathcal{E}$, since there is no minimal balanced collection that contains N. If $\mathcal{B}$ is not minimal, then there exists a $\mathcal{B}' \in \mathbb{B}^N_{\text{min}}$ such that $\mathcal{B}' \subsetneq \mathcal{B}$. More precisely, as every balanced collection is the union of minimal balanced collections there exists a $\mathcal{B}' \in \mathbb{B}^N_{\text{min}}$ such that $\mathcal{B}' \subsetneq \mathcal{B}$ and $N \backslash T \in \mathcal{B}'$.
First suppose there exists a $\mathcal{B}' \subsetneq \mathcal{B}$ with $N \backslash T \in \mathcal{B}'$ and $\beta' \in \Lambda^+(\mathcal{B}')$ such that $\beta'_{N \backslash T} < 1$. Take such a $\mathcal{B}'$. Then we obtain by definition of $\overline{\mathbb{B}}^N_{\text{min}}$ that $(\mathcal{B}' \backslash \{N \backslash T\}) \cup (\{T\}) \in \overline{\mathbb{B}}^N_{\text{min}} \subseteq \mathbb{E}^N_{\text{min}} \backslash \mathbb{B}^N_{\text{min}}$. Consequently, we have $\mathcal{B}' \backslash \{N \backslash T\} \cup \{T\} \subsetneq \mathcal{E}$, a contraction with the minimality of $\mathcal{E}$.
Next suppose that for every minimal balanced collection $\mathcal{C} \subsetneq \mathcal{B}$ with $N \backslash T \in \mathcal{C}$ it holds that $\beta^{\mathcal{C}}_{N \backslash T} = 1$. Take such a minimal balanced collection $\mathcal{C} \subsetneq \mathcal{B}$ with $N \backslash T \in \mathcal{C}$. We define a new collection $\mathcal{D} = \mathcal{C} \backslash \{N \backslash T\}$. Since $N \backslash T \notin \mathcal{D}$, we have $\mathcal{D} \subsetneq \mathcal{E}$. Also, $\sum_{S \in \mathcal{B}} \beta^{\mathcal{C}}_S e^S = e^N$ and therefore $\sum_{S \in \mathcal{D}} \beta^{\mathcal{C}}_S e^S = e^N - \beta^{\mathcal{C}}_{N \backslash T} e^{N \backslash T} = e^T$. Take $\delta_S = \beta^{\mathcal{C}}_S$ for every $S \in \mathcal{D}$ and $\delta_T = -1$, and we have $(1-a)\lambda^{\mathcal{E}} + a\delta \in \Lambda(\mathcal{E})$ for $a \in (0,1)$. This contradicts the minimality of $\mathcal{E}$, since by Theorem 4.3.7 $|\Lambda(\mathcal{E})| = 1$.
Hence, $\mathcal{B} \in \mathbb{B}^N_{\text{min}}$, $\mathcal{E} = \mathcal{B} \backslash \{N \backslash T\} \cup \{T\}$ and $\beta^{\mathcal{B}}_{N \backslash T} < 1$, so $\mathcal{E} \in \overline{\mathbb{B}}^N_{\text{min}}$. □

By definition, the grand coalition is not contained in any negative balanced collection. Hence, by Theorem 4.4.4 and 4.3.7, for every negative balanced collection the exact balanced weight vector is unique. Let $\mathcal{E} \in \overline{\mathbb{B}}^N_{\text{min}}$, take $\mathcal{B} \in \mathbb{B}^N_{\text{min}}$ and $U \in \mathcal{B}$ such that $\mathcal{E} = (\mathcal{B} \backslash \{U\}) \cup \{N \backslash U\}$. By Theorem 4.4.3, the negative weight is placed on $N \backslash U$. Since the exact balanced weight vector is unique, this implies that for every $\mathcal{E} \in \overline{\mathbb{B}}^N_{\text{min}}$ there is a unique collection $\mathcal{B} \in \mathbb{B}^N_{\text{min}}$ and a unique coalition $U \in \mathcal{B}$ such that $\mathcal{E} = (\mathcal{B} \backslash \{U\}) \cup \{N \backslash U\}$.

The third part of the partition consists of the *minimal subbalanced* collections. These collections consist of all minimal balanced collections of the proper subsets of the player set with at least two players, to which the coalition formed by all players of the subset and the coalition formed by all players of the original player set are added.

For every $M \subsetneq N$ such that $|M| \geq 2$, define

$$\widetilde{\mathbb{B}}^N_{\text{min}}(M) = \{\mathcal{B} \cup \{M, N\} \mid \mathcal{B} \in \mathbb{B}^M_{\text{min}}\},$$

Also, define

$$\widetilde{\mathbb{B}}^N_{\min} = \cup_{M \subsetneq N, |M| \geq 2} \widetilde{\mathbb{B}}^N_{\min}(M),$$

as the set of all minimal subbalanced collections.

Theorem 4.4.5 Let $\mathcal{E} \in \widetilde{\mathbb{B}}^N_{\min}$. Let $M \subsetneq N$ and $\mathcal{B} \in \mathbb{B}^M_{\min}$ be such that $\mathcal{E} = (\mathcal{B} \cup \{M, N\})$. Let $\lambda_S = \beta^{\mathcal{B}}_S$ for all $S \in \mathcal{B}$, $\lambda_M = -1$, $\lambda_N = 1$ and $\lambda_S = 0$ for all $S \in \mathcal{N} \backslash \mathcal{E}$. Then $\lambda \in \Lambda(\mathcal{E})$.

Proof: It is readily checked that $\sum_{S \in \mathcal{E}} \lambda_S e^S = \sum_{S \in \mathcal{B}} \beta^{\mathcal{B}}_S e^S - e^M + e^N = e^N$ and $\lambda_S > 0$ for all $S \in \mathcal{E} \backslash \{M\}$. Hence, $\lambda \in \Lambda(\mathcal{E})$. $\square$

Theorem 4.4.6

(i) $\widetilde{\mathbb{B}}^N_{\min} \subseteq \mathbb{E}^N_{\min} \backslash \mathbb{B}^N_{\min}$,

(ii) Let $\mathcal{E} \in \mathbb{E}^N_{\min} \backslash \mathbb{B}^N_{\min}$ and $N \in \mathcal{E}$. Then $\mathcal{E} \in \widetilde{\mathbb{B}}^N_{\min}$.

Proof:
(i) Let $\mathcal{E} \in \widetilde{\mathbb{B}}^N_{\min}$. Let $M \subsetneq N$ and $\mathcal{B} \in \mathbb{B}^M_{\min}$ be such that $\mathcal{E} = (\mathcal{B} \backslash \{M, N\})$. Let $\lambda_S = \beta^{\mathcal{B}}_S$ for all $S \in \mathcal{B}$, $\lambda_M = -1$, $\lambda_N = 1$ and $\lambda_S = 0$ for all $S \in \mathcal{N} \backslash \mathcal{E}$. Theorem 4.4.5 shows that $\lambda \in \Lambda(\mathcal{E})$, so $\mathcal{E} \in \mathbb{E}^N$.
Suppose $\mathcal{E} \notin \mathbb{E}^N_{\min}$. Take $\mathcal{D} \subsetneq \mathcal{E}$ such that $\mathcal{D} \in \mathbb{E}^N_{\min}$. We have $N \in \mathcal{D}$ since the players in $N \backslash M$ are not present in any other coalition in $\mathcal{E}$. This also implies that $\lambda^{\mathcal{D}}_N = 1$. As $\{N\} \notin \mathbb{E}^N$ we have $\sum_{S \in \mathcal{D} \backslash \{N\}} \lambda^{\mathcal{D}}_S = 0$. This means that there exists a $T \in \mathcal{D} \backslash \{N\}$ such that $\lambda^{\mathcal{D}}_T < 0$ and $S \subseteq T$ for all $S \in \mathcal{D} \backslash \{N\}$. We obtain $\mathcal{D} \backslash \{N, T\} \in \mathbb{B}^T$.
First, suppose $T = M$. Then $\mathcal{D} \subsetneq \mathcal{E}$ gives $\mathcal{D} \backslash \{N, M\} \subsetneq \mathcal{B}$ which contradicts $\mathcal{B} \in \mathbb{B}^M_{\min}$.
Second, suppose $T \neq M$. As $M \notin \mathcal{D}$, $\mathcal{D} \backslash \{N\} \subsetneq \mathcal{B}$. Define the weight vector δ such that $\delta_S = \lambda^{\mathcal{D}}_S$ for all $S \in \mathcal{D} \backslash \{N\}$ and $\delta_S = 0$ otherwise. Now, for small $\varepsilon > 0$ we have $\varepsilon \lambda^{\mathcal{D}} + \beta^{\mathcal{B}} \in \Lambda^+(\mathcal{B})$ which contradicts $\mathcal{B} \in \mathbb{B}^M_{\min}$ by Theorem 4.2.7.

(ii) By Theorem 4.3.9 we have $\lambda^{\mathcal{E}}_N = 1$. Take $T \in \mathcal{E}$ such that $\lambda^{\mathcal{E}}_T = -1$.

$\lvert N \rvert$	3	4	5
$\lvert \mathbb{E}^N_{\min} \rvert$	11	165	8572
$\lvert \mathbb{B}^N_{\min} \rvert$	5	41	1474
$\lvert \overline{\mathbb{B}}^N_{\min} \rvert$	3	98	6833
$\lvert \widetilde{\mathbb{B}}^N_{\min} \rvert$	3	26	265

Table 4.4.1: Number of collections in the three parts of the partition

We have $\sum_{S \in \mathcal{E} \backslash \{N\}} \lambda_S e^S = 0$ which yields $\sum_{S \in \mathcal{E} \backslash \{N,T\}} \lambda_S e^S = e^T$, and therefore $\mathcal{E} \backslash \{N, T\} \in \mathbb{B}^T$. If there exists a minimal balanced collection $\mathcal{B} \in \mathbb{B}^T_{\min}$ such that $\mathcal{B} \subsetneq \mathcal{E} \backslash \{T, N\}$, it is readily checked that $\mathcal{B} \cup \{T, N\}$ is an exact balanced collection, which contradicts our assumption of $\mathcal{E} \in \mathbb{E}^N_{\min}$. Hence, $\mathcal{E} \backslash \{N, T\} \in \mathbb{B}^T_{\min}$ and $\mathcal{E} \in \widetilde{\mathbb{B}}^N_{\min}$. $\square$

So, we established that $\widetilde{\mathbb{B}}^N_{\min}$, $\overline{\mathbb{B}}^N_{\min}$, and $\mathbb{B}^N_{\min}$ are contained in $\mathbb{E}^N_{\min}$. By definition, $N \notin \mathcal{E}$ for every $\mathcal{E} \in \overline{\mathbb{B}}^N_{\min}$ and $N \in \mathcal{E}$ for every $\mathcal{E} \in \widetilde{\mathbb{B}}^N_{\min}$. For $\mathcal{E} \in \mathbb{E}^N_{\min} \backslash \mathbb{B}^N_{\min}$, if $N \notin \mathcal{E}$ then $\mathcal{E} \in \overline{\mathbb{B}}^N_{\min}$, and if $N \in \mathcal{E}$ then $\mathcal{E} \in \widetilde{\mathbb{B}}^N_{\min}$. Hence, we obtain the following corollary from Theorem 4.4.1, Theorem 4.4.4 and Theorem 4.4.6.

Corollary 4.4.7 *The three sets $\widetilde{\mathbb{B}}^N_{\min}$, $\overline{\mathbb{B}}^N_{\min}$, and $\mathbb{B}^N_{\min}$ form a partition of $\mathbb{E}^N_{\min}$.*

The number of collections in the different parts of the partition are shown in Table 4.4.1. Appendix 4.A contains the minimal balanced, minimal subbalanced and minimal negative balanced collections for three and four players. To generate these collections we used the results by Peleg (1965), which provides an efficient and comprehensive algorithm for obtaining all minimal balanced collections. As we derived an explicit relation between minimal balanced collections on the one hand and minimal negative balanced collections and minimal subbalanced collections on the other hand (cf. Theorem 4.4.4 and Theorem 4.4.6), these collections can be constructed from the minimal balanced collections together with their respective weight vectors (cf. Theorem 4.4.3 and Theorem 4.4.5).

4.5 Sufficient conditions for exactness

As mentioned before, the class of minimal balanced collections is useful as one does not need other balanced collections to check whether a game is balanced. The class of minimal exact balanced weights exhibits the same feature: we show that we only need the minimal exact balanced collections to check whether a game is exact. To prove this, we first need the following lemma.

Lemma 4.5.1 *Let* $U \in \mathcal{N}$, $\mathcal{A} \subseteq \{S \in \mathcal{N} \mid S \subsetneq U\}$ *and* $\alpha \in \mathbb{R}_+^{\mathcal{N}}$ *such that* $\sum_{S \in \mathcal{A}} \alpha_S e^S = e^U$, $\alpha_S > 0$ *for every* $S \in \mathcal{A}$ *and* $\alpha_S = 0$ *for every* $S \in 2^U \backslash (\mathcal{A} \cup \{U\})$. *Let* $v \in \cap_{\mathcal{E} \in \widetilde{\mathbb{B}}^N_{\min}(U)} V(\mathcal{E})$. *Then* $\sum_{S \in \mathcal{A}} \alpha_S v(S) \leq v(U)$.

Proof: Take (U, v^U) such that $v^U(S) = v(S)$ for every $S \in 2^U$, and take $\alpha^U \in \mathbb{R}^{2^U}$ such that $\alpha^U_S = \alpha_S$ for every $S \in 2^U$. Since $\sum_{S \in \mathcal{A}} \alpha^U_S e^S = e^U$, we have $\mathcal{A} \in \mathbb{B}^U$ with α^U the corresponding balanced weight vector. By Theorem 4.2.6, this means that there exist $\mathcal{A}_1, ..., \mathcal{A}_m \in \mathbb{B}^U_{\min}$ with balanced weight vectors $\alpha^1, ..., \alpha^m \in \mathbb{R}^{2^U}$ such that if

$$\sum_{S \in \mathcal{A}_i} \alpha^i_S v^U(S) \leq v^U(U) \text{ for every } i \in \{1, ..., m\},$$

then

$$\sum_{S \in \mathcal{A}} \alpha^U_S v^U(S) \leq v^U(U).$$

Note that $\sum_{S \in \mathcal{A}} \alpha^U_S v^U(S) \leq v^U(U)$ directly implies $\sum_{S \in \mathcal{A}} \alpha_S v(S) \leq v(U)$. We show that for every $i \in \{1, ..., m\}$, $\sum_{S \in \mathcal{A}_i} \alpha^i_S v^U(S) \leq v^U(U)$ follows from $v \in \cap_{\mathcal{E} \in \widetilde{\mathbb{B}}^N_{\min}(U)} V(\mathcal{E})$.
Take $i \in \{1, ..., m\}$ and $\mathcal{C}_i = \mathcal{A}_i \cup \{U, N\}$. By definition, we have $\mathcal{C}_i \in \widetilde{\mathbb{B}}^N_{\min}(U)$ with $\lambda^{\mathcal{C}_i}_S = \alpha^i_S$ for every $S \in \mathcal{C}_i$, $\lambda^{\mathcal{C}_i}_U = -1$ and $\lambda^{\mathcal{C}_i}_N = 1$. We obtain

$$\begin{aligned} \sum_{S \in \mathcal{A}_i} \alpha^i_S v^U(S) &= \sum_{S \in \mathcal{A}_i} \alpha^i_S v(S) \\ &= \sum_{S \in \mathcal{C}_i} \lambda^{\mathcal{C}_i}_S v(S) + v(U) - v(N) \\ &\leq v(N) + v(U) - v(N) \\ &= v^U(U), \end{aligned}$$

where the inequality follows from $v \in \cap_{\mathcal{E} \in \widetilde{\mathbb{B}}^N_{\min}(U)} V(\mathcal{E})$. Hence, for every $i \in \{1, ..., m\}$, $\sum_{S \in \mathcal{A}_i} \alpha^i_S v^U(S) \leq v^U(U)$ and therefore $\sum_{S \in \mathcal{A}} \alpha_S v(S) \leq v(U)$. $\square$

Theorem 4.5.2 Let $v \in V(\mathcal{E})$ for all $\mathcal{E} \in \mathbb{E}^N_{\min}$. Then $v \in V$.

Proof: In fact, we prove the following statement. Let $\mathcal{D} \in \mathbb{E}^N \backslash \mathbb{E}^N_{\min}$, and assume $\cap_{\mathcal{E} \in \mathbb{E}^N_{\min}} V(\mathcal{E}) \subseteq V(\mathcal{A})$ for every $\mathcal{A} \subsetneq \mathcal{D}$. Then $\cap_{\mathcal{E} \in \mathbb{E}^N_{\min}} V(\mathcal{E}) \subseteq V(\mathcal{D})$. Clearly, this holds for every $\mathcal{D} \in \mathbb{E}^N \backslash \mathbb{E}^N_{\min}$ with $|\mathcal{D}| = 0$. By induction on $|\mathcal{D}|$ we then obtain $\cap_{\mathcal{E} \in \mathbb{E}^N_{\min}} V(\mathcal{E}) = \cap_{\mathcal{E} \in \mathbb{E}^N} V(\mathcal{E}) = V$.

To show that $\cap_{\mathcal{E} \in \mathbb{E}^N_{\min}} V(\mathcal{E}) \subseteq V(\mathcal{D})$, we show $\cap_{\mathcal{E} \in \mathbb{E}^N_{\min}} V(\mathcal{E}) \subseteq V(\delta)$ for every $\delta \in \Lambda(\mathcal{D})$. Let $\delta \in \Lambda(\mathcal{D})$.
First, assume $\delta \in \Lambda^+(\mathcal{D})$. Then Theorem 4.4.1 and Theorem 4.2.6 imply that $v \in V(\delta)$.
Second, assume that $\delta \notin \Lambda^+(\mathcal{D})$. Take $U \in \mathcal{D}$ such that $\delta_U < 0$.
If $U = N$, then define $\mathcal{C} = \mathcal{D} \backslash \{N\}$ and $\gamma_S = \frac{\delta_S}{1-\delta_N}$ for all $S \in \mathcal{C}$ and $\gamma_S = 0$ for all $S \in \mathcal{N} \backslash \mathcal{C}$. We have $\gamma \in \Lambda^+(\mathcal{C})$ and $\mathcal{C} \in \mathbb{B}^N$. Note that $v \in V(\delta)$ is directly implied by $v \in V(\gamma)$. Hence, in the remainder we will assume that $U \neq N$.
Since $\mathcal{D} \notin \mathbb{E}^N_{\min}$, we can take $\mathcal{A} \in \mathbb{E}^N_{\min}$ such that $\mathcal{A} \subsetneq \mathcal{D}$.
If $\mathcal{A} \in \mathbb{E}^N_{\min} \backslash \mathbb{B}^N_{\min}$, then take $T \in \mathcal{A}$ such that $\lambda^{\mathcal{A}}_T < 0$. If $\mathcal{A} \in \mathbb{B}^N_{\min}$, define $T = \emptyset$.
Define $a = \min\{\frac{\delta_S}{\lambda^{\mathcal{A}}_S} \mid S \in \mathcal{A} \backslash \{T, U\}\}$. We first show that $a \leq 1$.
Suppose on the contrary that $a > 1$. As $\delta_S > \lambda^{\mathcal{A}}_S$ for every $S \in \mathcal{D} \backslash \{U\}$, we have for $i \in N \backslash U$ that

$$\sum_{\substack{S \in \mathcal{D}, \\ S \ni i}} \delta_S = \sum_{\substack{S \in \mathcal{D} \backslash \{U\}, \\ S \ni i}} \delta_S > \sum_{\substack{S \in \mathcal{D} \backslash \{U\}, \\ S \ni i}} \lambda^{\mathcal{A}}_S = 1,$$

a contradiction.
We discriminate between two cases:

- $a = 1$. If $T \backslash U \neq \emptyset$, then for $i \in T \backslash U$ it holds that

$$\sum_{\substack{S \in \mathcal{D}, \\ S \ni i}} \delta_S = \sum_{\substack{S \in \mathcal{D} \backslash \{T,U\}, \\ S \ni i}} \delta_S + \delta_T > \sum_{\substack{S \in \mathcal{D} \backslash \{T,U\}, \\ S \ni i}} \lambda^{\mathcal{A}}_S + \lambda^{\mathcal{A}}_T = 1,$$

  which cannot hold. So, $T \backslash U = \emptyset$. If $\delta_U = \lambda^{\mathcal{A}}_U$ then $\delta_R > \lambda^{\mathcal{A}}_R$ for some $R \in \mathcal{A}$ would give

$$\sum_{\substack{S \in \mathcal{D}, \\ S \ni i}} \delta_S > \sum_{\substack{S \in \mathcal{D}, \\ S \ni i}} \lambda^{\mathcal{A}}_S = 1,$$

for $i \in R$. Hence, $\delta_S = \lambda_S$ for every $S \in \mathcal{D}$ but this contradicts $\mathcal{A} \subsetneq \mathcal{D}$.

So, we conclude $T \backslash U = \emptyset$ and $\delta_U < \lambda_U^{\mathcal{A}}$, implying $\lambda_U^{\mathcal{A}} - \delta_U > 0$. Define $\kappa_S = \frac{\delta_S - \lambda_S^{\mathcal{A}}}{\lambda_U^{\mathcal{A}} - \delta_U}$ for all $S \in \mathcal{D} \backslash \{U\}$, and $\kappa_S = 0$ for all $S \in \mathcal{N} \backslash (\mathcal{D} \backslash \{U\})$. Take $\mathcal{K} = \{S \in \mathcal{D} \backslash \{U\} \mid \kappa_S \neq 0\}$. Note that $\kappa_S > 0$ for all $S \in \mathcal{K}$ and

$$\begin{aligned}
\sum_{S \in \mathcal{K}} \kappa_S e^S &= \frac{1}{\lambda_U^{\mathcal{A}} - \delta_U} \sum_{S \in \mathcal{D} \backslash \{U\}} (\delta_S - \lambda_S^{\mathcal{A}}) e^S \\
&= \frac{1}{\lambda_U^{\mathcal{A}} - \delta_U} \left(\sum_{S \in \mathcal{D} \backslash \{U\}} \delta_S e^S - \sum_{S \in \mathcal{A} \backslash \{U\}} \lambda_S^{\mathcal{A}} e^S \right) \\
&= \frac{1}{\lambda_U^{\mathcal{A}} - \delta_U} (e^N - \delta_U e^U - e^N + \lambda_U^{\mathcal{A}} e^U) \\
&= e^U.
\end{aligned}$$

Hence, $\mathcal{K} \subseteq \{S \in \mathcal{N} \mid S \subsetneq U\}$, $\sum_{S \in \mathcal{K}} \kappa_S e^S = e^U$, $\kappa_S > 0$ for every $S \in \mathcal{A}$ and $\kappa_S = 0$ for every $S \in 2^U \backslash (\mathcal{K} \cup \{U\})$. So, we can apply Lemma 4.5.1. Therefore

$$\begin{aligned}
\sum_{S \in \mathcal{D}} \delta_S v(S) &= (\lambda_U^{\mathcal{A}} - \delta_U) \sum_{S \in \mathcal{D}} \frac{\delta_S - \lambda_S^{\mathcal{A}}}{\lambda_U^{\mathcal{A}} - \delta_U} v(S) \\
&\quad + \sum_{S \in \mathcal{D}} \lambda_S^{\mathcal{A}} v(S) \\
&= (\lambda_U^{\mathcal{A}} - \delta_U) \left(\sum_{S \in \mathcal{K}} \kappa_S v(S) - v(U) \right) \\
&\quad + \sum_{S \in \mathcal{A}} \lambda_S^{\mathcal{A}} v(S) \\
&\leq (\lambda_U^{\mathcal{A}} - \delta_U)(v(U) - v(U)) + v(N) \\
&= v(N),
\end{aligned}$$

where the inequality follows from Lemma 4.5.1 and $v \in V(\lambda^{\mathcal{A}})$. Hence, $(\cap_{\mathcal{E} \in \widetilde{\mathbb{B}}_{\min}^N(U)} V(\mathcal{E})) \cap V(\mathcal{A}) \subseteq V(\delta)$. Since $\widetilde{\mathbb{B}}_{\min}^N(U) \subseteq \mathbb{E}_{\min}^N$ and $\mathcal{A} \in \mathbb{E}_{\min}^N$, we can now conclude that $\cap_{\mathcal{E} \in \mathbb{E}_{\min}^N} V(\mathcal{E}) \subseteq V(\delta)$.

- $a < 1$. We define $\kappa_S = \frac{1}{1-a}\delta_S - \frac{a}{1-a}\lambda_S^{\mathcal{A}}$ for all $S \in \mathcal{D}$, $\kappa_S = 0$ for $S \in \mathcal{N}\backslash\mathcal{D}$ and define $\mathcal{K} = \{S \in \mathcal{D} \mid \kappa_S \neq 0\}$. By definition of a, $\mathcal{K} \subsetneq \mathcal{D}$. Note that $\mathcal{K} \in \mathbb{E}^N$ and $\kappa \in \Lambda(\mathcal{K})$, as $\kappa_S > 0$ for all $S \in \mathcal{K}\backslash\{U\}$, $\kappa_U < 0$, and

$$\begin{aligned}\sum_{S\in\mathcal{K}} \kappa_S e^S &= \sum_{S\in\mathcal{D}} (\frac{1}{1-a}\delta_S - \frac{a}{1-a}\lambda_S^{\mathcal{A}})e^S \\ &= \frac{1}{1-a}\sum_{S\in\mathcal{D}} \delta_S e^S - \frac{a}{1-a}\sum_{S\in\mathcal{A}} \lambda_S^{\mathcal{A}} e^S \\ &= e^N.\end{aligned}$$

It is now easily seen that $V(\kappa) \cap V(\lambda^{\mathcal{A}}) \subseteq V(\delta)$, as

$$\sum_{S\in\mathcal{D}} \delta_S v(S) = (1-a)\sum_{S\in\mathcal{K}} \kappa_S v(S) + a\sum_{S\in\mathcal{A}} \lambda_S^{\mathcal{A}} v(S) \leq v(N).$$

Hence, $V(\mathcal{K}) \cap V(\mathcal{A}) \subseteq V(\delta)$. Since $\mathcal{K} \subseteq \mathcal{D}$ and $\mathcal{A} \in \mathbb{E}^N_{\min}$, by induction we can now conclude that $\cap_{\mathcal{E}\in\mathbb{E}^N_{\min}} V(\mathcal{E}) \subseteq V(\delta)$.

□

As Theorem 4.2.8 states, the set of balanced games can not be characterized by a subset of the minimal balanced collections. However, the set of exact games can be characterized by a subset of the minimal exact balanced collections, as there exist minimal exact balanced collections that are redundant. The following example illustrates this.

Example 4.5.3 Consider the minimal exact balanced collections $\mathcal{B} = \{\{1,2\},\{1,3\},\{2,3\}\}$ with weight vector $\beta^{\mathcal{B}}$ such that $\beta^{\mathcal{B}}_{\{1,2\}} = \beta^{\mathcal{B}}_{\{1,3\}} = \beta^{\mathcal{B}}_{\{2,3\}} = \frac{1}{2}$, $\mathcal{C} = \{\{1\},\{2,3\}\}$ with weights $\beta^{\mathcal{C}}_{\{1\}} = \beta^{\mathcal{C}}_{\{2,3\}} = 1$ and $\mathcal{E} = \{\{1,2\},\{1,3\},\{1\}\}$ with $\lambda^{\mathcal{E}}_{\{1,2\}} = \lambda^{\mathcal{E}}_{\{1,3\}} = 1$ and $\lambda^{\mathcal{E}}_{\{1\}} = -1$. We have $V(\mathcal{C}) \cap V(\mathcal{E}) \subseteq V(\mathcal{B})$, since $\beta^{\mathcal{B}} = \frac{1}{2}\beta^{\mathcal{C}} + \frac{1}{2}\lambda^{\mathcal{E}}$. ◁

The question arises which minimal exact balanced collections we can discard. It turns out that for $|N| \geq 3$, $\cap_{\mathcal{E}\in\mathbb{E}^N_{\min}\backslash\mathbb{B}^N_{\min}} V(\mathcal{E}) \subseteq V$. So, we can omit all the minimal balanced conditions. To show this, we first introduce a lemma to construct particular members of $\mathbb{E}^N_{\min}$.

Lemma 4.5.4 *Let $|N| \geq 3$ and take $S \in \mathcal{N}$ and $T \in \mathcal{N}$ such that $S \cap T = \emptyset$.*

(i) If $S \cup T = N$, $|T| \geq 2$ and $i \in T$, then $\{S \cup \{i\}, T, \{i\}\} \in \mathbb{E}^N_{\min} \backslash \mathbb{B}^N_{\min}$.

(ii) If $S \cup T \neq N$, then $\{S, T, S \cup T, N\} \in \mathbb{E}^N_{\min} \backslash \mathbb{B}^N_{\min}$.

Proof:
(i) The collection $\{S \cup \{i\}, T, N \backslash \{i\}\}$ is minimal balanced with weight vector λ such that $\lambda_{S\cup\{i\}} = \lambda_T = \lambda_{N\backslash\{i\}} = \frac{1}{2}$. By definition of $\overline{\mathbb{B}}^N_{\min}$, we have $\{S \cup \{i\}, T, \{i\}\} \in \overline{\mathbb{B}}^N_{\min}$. By Theorem 4.4.4 this means that $\{S \cup \{i\}, T, \{i\}\} \in \mathbb{E}^N_{\min} \backslash \mathbb{B}^N_{\min}$.

(ii) The collection $\{S, T\}$ is minimal balanced for player set $S \cup T$. By definition of $\widetilde{\mathbb{B}}^N_{\min}(S \cup T)$, we have $\{S, T, S \cup T, N\} \in \widetilde{\mathbb{B}}^N_{\min}(S \cup T)$. By Theorem 4.4.6 this means that $\{S, T, S \cup T, N\} \in \mathbb{E}^N_{\min} \backslash \mathbb{B}^N_{\min}$. $\square$

Theorem 4.5.5 Let $v \in V(\mathcal{E})$ for every $\mathcal{E} \in \mathbb{E}^N_{\min} \backslash \mathbb{B}^N_{\min}$ and $|N| \geq 3$. Then $v \in V$.

Proof: Let $\mathcal{B} \in \mathbb{B}^N_{\min}$. We show that $\cap_{\mathcal{E} \in \mathbb{E}^N_{\min} \backslash \mathbb{B}^N_{\min}} V(\mathcal{E}) \subseteq V(\mathcal{B})$. First, consider the case where $\mathcal{B}$ is a partition.
Assume $\mathcal{B} = \{S, T\}$ for some $S, T \in \mathcal{N}$. Note that $\beta^{\mathcal{B}}_S = \beta^{\mathcal{B}}_T = 1$. Without loss of generality, we assume $|S| \leq |T|$. Take $i \in T$, $\mathcal{A} = \{\{i\}, S, S \cup \{i\}, N\}$ with $\lambda^{\mathcal{A}}_{\{i\}} = \lambda^{\mathcal{A}}_S = \lambda^{\mathcal{A}}_N = 1$ and $\lambda^{\mathcal{A}}_{S\cup\{i\}} = -1$ and $\mathcal{D} = \{S \cup \{i\}, T, \{i\}\}$ with $\lambda^{\mathcal{D}}_{S\cup\{i\}} = \lambda^{\mathcal{D}}_T = 1$ and $\lambda^{\mathcal{D}}_{\{i\}} = -1$. By Lemma 4.5.4, $\mathcal{A} \in \mathbb{E}^N_{\min} \backslash \mathbb{B}^N_{\min}$ and $\mathcal{D} \in \mathbb{E}^N_{\min} \backslash \mathbb{B}^N_{\min}$. Now $V(\mathcal{A}) \cap V(\mathcal{D}) \subseteq V(\mathcal{B})$ as

$$
\begin{aligned}
\sum_{U \in \mathcal{B}} \beta^{\mathcal{B}}_U v(U) &= v(S) + v(T) \\
&= (v(\{i\}) + v(S) - v(S \cup \{i\}) + v(N)) \\
&\quad + \ (v(S \cup \{i\}) + v(T) - v(\{i\})) \\
&\quad - v(N) \\
&= \sum_{U \in \mathcal{A}} \lambda^{\mathcal{A}}_U v(U) + \sum_{U \in \mathcal{D}} \lambda^{\mathcal{D}}_U v(U) - v(N) \\
&\leq v(N),
\end{aligned}
$$

where the inequality follows from $v \in V(\mathcal{A})$ and $v \in V(\mathcal{D})$.

We show that for every partition $\mathcal{B}$ with $|\mathcal{B}| \geq 3$ there exists a partition $\mathcal{C}$ such that $|\mathcal{C}| < |\mathcal{B}|$ and $\cap_{\mathcal{E}\in\mathbb{E}^N_{\min}\backslash\mathbb{B}^N_{\min}} V(\mathcal{E}) \cap V(\mathcal{C}) \subseteq V(\mathcal{B})$. This suffices to show that $\cap_{\mathcal{E}\in\mathbb{E}^N_{\min}\backslash\mathbb{B}^N_{\min}} V(\mathcal{E}) \subseteq V(\mathcal{B})$ for every partition $\mathcal{B} \in \mathbb{B}^N_{\min}$.
Assume that $\mathcal{B}$ is a partition of the player set N, with $|\mathcal{B}| \geq 3$. Take $S \in \mathcal{B}$ and $T \in \mathcal{B}$ with $S \neq T$. Define $\mathcal{A} = \{S, T, S \cup T, N\}$ with $\lambda^{\mathcal{A}}_S = \lambda^{\mathcal{A}}_T = \lambda^{\mathcal{A}}_N = 1$ and $\lambda^{\mathcal{A}}_{S\cup T} = -1$. By Lemma 4.5.4 we have $\mathcal{A} \in \mathbb{E}^N_{\min}\backslash\mathbb{B}^N_{\min}$. Define $\mathcal{D} = (\mathcal{B}\backslash\{S,T\}) \cup \{S \cup T\}$ and $\delta \in \Lambda(\mathcal{D})$ such that $\delta_S = 1$ for all $S \in \mathcal{D}$. Note that $\mathcal{D}$ is a partition and $|\mathcal{D}| < |\mathcal{B}|$. Now $V(\mathcal{A}) \cap V(\mathcal{D}) \subseteq V(\mathcal{B})$ as

$$\begin{aligned}
\sum_{U\in\mathcal{B}} \beta^{\mathcal{B}}_U v(U) &= \sum_{U\in\mathcal{B}\backslash\{S,T\}} \beta^{\mathcal{B}}_U v(U) + v(S) + v(T) \\
&= \sum_{U\in\mathcal{B}\backslash\{S,T\}} \beta^{\mathcal{B}}_U v(U) + v(S\cup T) \\
&\quad + v(S) + v(T) - v(S\cup T) + v(N) \\
&\quad - v(N) \\
&= \sum_{U\in\mathcal{D}} \lambda^{\mathcal{D}}_U v(U) + \sum_{U\in\mathcal{A}} \lambda^{\mathcal{A}}_U v(U) - v(N) \\
&\leq v(N),
\end{aligned}$$

where the inequality follows from $v \in V(\mathcal{A})$ and $v \in V(\mathcal{D})$.
Second, consider the case where $\mathcal{B}$ is not a partition. Take $T \in \mathcal{B}$ such that $\beta^{\mathcal{B}}_T < 1$. As $\mathcal{B}$ is not a partition, such a coalition exists and $N\backslash T \notin \mathcal{B}$. Define $\mathcal{C} = \{T, N\backslash T\}$ and $\mathcal{D} = (\mathcal{B}\backslash\{T\}) \cup \{N\backslash T\}$ with $\delta_S = \frac{\beta^{\mathcal{B}}_S}{1-\beta^{\mathcal{B}}_T}$ for all $S \in \mathcal{B}\backslash\{T\}$ and $\delta_{N\backslash T} = -\frac{\beta^{\mathcal{B}}_T}{1-\beta^{\mathcal{B}}_T}$. We have already shown that $\cap_{\mathcal{E}\in\mathbb{E}^N_{\min}\backslash\mathbb{B}^N_{\min}} V(\mathcal{E}) \subseteq V(\mathcal{C})$. Furthermore, by Theorem 4.4.4 we know that $\mathcal{D} \in \mathbb{E}^N_{\min}\backslash\mathbb{B}^N_{\min}$. Now $V(\mathcal{C}) \cap V(\mathcal{D}) \subseteq V(\mathcal{B})$ as

$$\begin{aligned}
\sum_{S\in\mathcal{B}} \beta^{\mathcal{B}}_S v(S) &= \sum_{S\in\mathcal{B}\backslash\{T\}} \beta^{\mathcal{B}}_S v(S) + \beta^{\mathcal{B}}_T v(T) \\
&= (1-\beta^{\mathcal{B}}_T)\left(\sum_{S\in\mathcal{B}\backslash\{T\}} \frac{\beta^{\mathcal{B}}_S}{1-\beta^{\mathcal{B}}_T} v(S) - \frac{\beta^{\mathcal{B}}_T}{1-\beta^{\mathcal{B}}_T} v(N\backslash T)\right) \\
&\quad + \beta^{\mathcal{B}}_T (v(T) + v(N\backslash T)) \\
&= (1-\beta^{\mathcal{B}}_T) \sum_{S\in\mathcal{D}} \delta_S v(S) + \beta_T \sum_{S\in\mathcal{C}} \beta^{\mathcal{C}}_S v(S) \\
&\leq v(N),
\end{aligned}$$

where the inequality follows from $v \in V(\mathcal{C})$ and $v \in V(\mathcal{D})$. So $\cap_{\mathcal{E}\in\mathbb{E}^N_{\min}\backslash\mathbb{B}^N_{\min}} V(\mathcal{E}) \subseteq \cap_{\mathcal{B}\in\mathbb{B}^N_{\min}} V(\mathcal{B})$.

Therefore, $\cap_{\mathcal{E}\in\mathbb{E}^N_{\min}\setminus\mathbb{B}^N_{\min}} V(\mathcal{E}) = \cap_{\mathcal{E}\in\mathbb{E}^N_{\min}} V(\mathcal{E}) = \cap_{\mathcal{E}\in\mathbb{E}^N} V(\mathcal{E}) = V$. So, $v \in V$ if and only if $v \in v(\mathcal{E})$ for all $\mathcal{E} \in \mathbb{E}^N_{\min}\setminus\mathbb{B}^N_{\min}$. $\square$

We have shown that the class of minimal balanced collections is redundant to verify that a game is exact. However, as the following example demonstrates, there exists an even smaller subclass of the class of minimal exact balanced collections that still ensures exactness of the game.

Example 4.5.6 Let $N = \{1,2,3,4\}$. Consider the minimal exact balanced collections $\mathcal{A} = \{\{2\},\{1,4\},\{1,2,4\},N\}$, $\mathcal{D} = \{\{1,2,4\},\{1,2,3\},\{1,2\}\}$ and $\mathcal{E} = \{\{2\},\{1,2\},\{1,4\},\{1,2,3\}\}$. From

$$v(\{2\}) + v(\{1,4\}) - v(\{1,2,4\}) + v(N) \leq v(N),$$

and

$$v(\{1,2,4) + v(\{1,2,3\}) - v(\{1,2\}) \leq v(N),$$

we have that

$$v(\{2\}) + v(\{1,4\}) + v(\{1,2,3\}) - v(\{1,2\}) \leq v(N).$$

This implies that $V(\mathcal{A}) \cap V(\mathcal{D}) \subseteq V(\mathcal{E})$, so $\mathcal{E}$ is redundant. $\triangleleft$

Further research on the topic could possibly establish a characterization of a subclass of minimal exact balanced collections where no collection can be left out while still guaranteing exactness.

4.A Minimal exact balanced collections

4.A.1 $N = \{1, 2, 3\}$

Minimal balanced

Collections			Weights		
{1}	{2}	{3}	1	1	1
{3}	{1,2}		1	1	
{2}	{1,3}		1	1	
{1}	{2,3}		1	1	
{1,2}	{1,3}	{2,3}	1/2	1/2	1/2

Minimal negative balanced

Collections			Weights		
{1}	{1,2}	{1,3}	-1	1	1
{2}	{1,2}	{2,3}	-1	1	1
{3}	{1,3}	{2,3}	-1	1	1

Minimal subbalanced

Collections				Standardized weights			
{1}	{2}	{1,2}	{1,2,3}	1	1	-1	1
{1}	{3}	{1,3}	{1,2,3}	1	1	-1	1
{2}	{3}	{2,3}	{1,2,3}	1	1	-1	1

4.A.2 $N = \{1, 2, 3, 4\}$

Minimal balanced Collections				Weights			
{1}	{2,3,4}			1	1		
{2}	{1,3,4}			1	1		
{3}	{1,2,4}			1	1		
{4}	{1,2,3}			1	1		
{1,2}	{3,4}			1	1		
{1,3}	{2,4}			1	1		
{1,4}	{2,3}			1	1		
{1}	{2}	{3,4}		1	1	1	
{1}	{3}	{2,4}		1	1	1	
{1}	{4}	{2,3}		1	1	1	
{2}	{3}	{1,4}		1	1	1	
{2}	{4}	{1,3}		1	1	1	
{3}	{4}	{1,2}		1	1	1	
{1,2}	{1,3,4}	{2,3,4}		1/2	1/2	1/2	
{1,3}	{1,2,4}	{2,3,4}		1/2	1/2	1/2	
{1,4}	{1,2,3}	{2,3,4}		1/2	1/2	1/2	
{2,3}	{1,2,4}	{1,3,4}		1/2	1/2	1/2	
{2,4}	{1,2,3}	{1,3,4}		1/2	1/2	1/2	
{3,4}	{1,2,3}	{1,2,4}		1/2	1/2	1/2	
{1}	{2}	{3}	{4}	1	1	1	1
{1}	{2,3}	{2,4}	{3,4}	1	1/2	1/2	1/2
{1}	{2,3}	{2,4}	{1,3,4}	1/2	1/2	1/2	1/2
{1}	{2,3}	{3,4}	{1,2,4}	1/2	1/2	1/2	1/2
{1}	{2,4}	{3,4}	{1,2,3}	1/2	1/2	1/2	1/2
{2}	{1,3}	{1,4}	{3,4}	1	1/2	1/2	1/2
{2}	{1,3}	{1,4}	{2,3,4}	1/2	1/2	1/2	1/2
{2}	{1,3}	{3,4}	{1,2,4}	1/2	1/2	1/2	1/2
{2}	{1,4}	{3,4}	{1,2,3}	1/2	1/2	1/2	1/2
{3}	{1,2}	{1,4}	{2,4}	1	1/2	1/2	1/2

Minimal balanced (continued)

Collections				Weights			
{3}	{1,2}	{1,4}	{2,3,4}	1/2	1/2	1/2	1/2
{3}	{1,2}	{2,4}	{1,3,4}	1/2	1/2	1/2	1/2
{3}	{1,4}	{2,4}	{1,2,3}	1/2	1/2	1/2	1/2
{4}	{1,2}	{1,3}	{2,3}	1	1/2	1/2	1/2
{4}	{1,2}	{1,3}	{2,3,4}	1/2	1/2	1/2	1/2
{4}	{1,2}	{2,3}	{1,3,4}	1/2	1/2	1/2	1/2
{4}	{1,3}	{2,3}	{1,2,4}	1/2	1/2	1/2	1/2
{1,2}	{1,3}	{1,4}	{2,3,4}	1/3	1/3	1/3	2/3
{1,2}	{2,3}	{2,4}	{1,3,4}	1/3	1/3	1/3	2/3
{1,3}	{2,3}	{3,4}	{1,2,4}	1/3	1/3	1/3	2/3
{1,4}	{2,4}	{3,4}	{1,2,3}	1/3	1/3	1/3	2/3
{1,2,3}	{1,2,4}	{1,3,4}	{2,3,4}	1/3	1/3	1/3	1/3

Minimal negative balanced

Collections				Weights			
{1}	{1,2}	{1,3,4}		-1	1	1	
{1}	{1,3}	{1,2,4}		-1	1	1	
{1}	{1,4}	{1,2,3}		-1	1	1	
{2}	{1,2}	{2,3,4}		-1	1	1	
{2}	{2,3}	{1,2,4}		-1	1	1	
{2}	{2,4}	{1,2,3}		-1	1	1	
{3}	{1,3}	{2,3,4}		-1	1	1	
{3}	{2,3}	{1,3,4}		-1	1	1	
{3}	{3,4}	{1,2,3}		-1	1	1	
{4}	{1,4}	{2,3,4}		-1	1	1	
{4}	{2,4}	{1,3,4}		-1	1	1	
{4}	{3,4}	{1,2,4}		-1	1	1	
{1,2}	{1,2,3}	{1,2,4}		-1	1	1	
{1,3}	{1,2,3}	{1,3,4}		-1	1	1	
{1,4}	{1,2,4}	{1,3,4}		-1	1	1	
{2,3}	{1,2,3}	{2,3,4}		-1	1	1	
{2,4}	{1,2,4}	{2,3,4}		-1	1	1	
{3,4}	{1,3,4}	{2,3,4}		-1	1	1	
{1}	{2}	{1,3}	{1,4}	-1	1	1	1
{1}	{2}	{2,3}	{2,4}	1	-1	1	1
{1}	{3}	{1,2}	{1,4}	-1	1	1	1
{1}	{3}	{2,3}	{3,4}	1	-1	1	1
{1}	{4}	{1,2}	{1,3}	-1	1	1	1
{1}	{4}	{2,4}	{3,4}	1	-1	1	1
{1}	{1,2}	{1,3}	{1,4}	-2	1	1	1
{1}	{1,2}	{2,3}	{2,4}	2	-1	1	1
{1}	{1,2}	{2,3}	{1,2,4}	1	-1	1	1
{1}	{1,2}	{2,4}	{1,2,3}	1	-1	1	1
{1}	{1,3}	{2,3}	{3,4}	2	-1	1	1
{1}	{1,3}	{2,3}	{1,3,4}	1	-1	1	1

Minimal negative balanced (continued)

Collections				Weights			
{1}	{1,3}	{3,4}	{1,2,3}	1	-1	1	1
{1}	{1,4}	{2,4}	{3,4}	2	-1	1	1
{1}	{1,4}	{2,4}	{1,3,4}	1	-1	1	1
{1}	{1,4}	{3,4}	{1,2,4}	1	-1	1	1
{1}	{1,2,3}	{1,2,4}	{1,3,4}	- 1/2	1/2	1/2	1/2
{2}	{3}	{1,2}	{2,4}	-1	1	1	1
{2}	{3}	{1,3}	{3,4}	1	-1	1	1
{2}	{4}	{1,2}	{2,3}	-1	1	1	1
{2}	{4}	{1,4}	{3,4}	1	-1	1	1
{2}	{1,2}	{1,3}	{1,4}	2	-1	1	1
{2}	{1,2}	{1,3}	{1,2,4}	1	-1	1	1
{2}	{1,2}	{1,4}	{1,2,3}	1	-1	1	1
{2}	{1,2}	{2,3}	{2,4}	-2	1	1	1
{2}	{1,3}	{2,3}	{3,4}	2	1	-1	1
{2}	{1,3}	{2,3}	{2,3,4}	1	1	-1	1
{2}	{1,4}	{2,4}	{3,4}	2	1	-1	1
{2}	{1,4}	{2,4}	{2,3,4}	1	1	-1	1
{2}	{2,3}	{3,4}	{1,2,3}	1	-1	1	1
{2}	{2,4}	{3,4}	{1,2,4}	1	-1	1	1
{2}	{1,2,3}	{1,2,4}	{2,3,4}	- 1/2	1/2	1/2	1/2
{3}	{4}	{1,3}	{2,3}	-1	1	1	1
{3}	{4}	{1,4}	{2,4}	1	-1	1	1
{3}	{1,2}	{1,3}	{1,4}	2	1	-1	1
{3}	{1,2}	{1,3}	{1,3,4}	1	1	-1	1
{3}	{1,2}	{2,3}	{2,4}	2	1	-1	1
{3}	{1,2}	{2,3}	{2,3,4}	1	1	-1	1
{3}	{1,3}	{1,4}	{1,2,3}	1	-1	1	1
{3}	{1,3}	{2,3}	{3,4}	-2	1	1	1
{3}	{1,4}	{2,4}	{3,4}	2	1	1	-1
{3}	{1,4}	{3,4}	{2,3,4}	1	1	-1	1
{3}	{2,3}	{2,4}	{1,2,3}	1	-1	1	1
{3}	{2,4}	{3,4}	{1,3,4}	1	1	-1	1

Minimal negative balanced (continued)

Collections				Weights			
{3}	{1,2,3}	{1,3,4}	{2,3,4}	- 1/2	1/2	1/2	1/2
{4}	{1,2}	{1,3}	{1,4}	2	1	1	-1
{4}	{1,2}	{1,4}	{1,3,4}	1	1	-1	1
{4}	{1,2}	{2,3}	{2,4}	2	1	1	-1
{4}	{1,2}	{2,4}	{2,3,4}	1	1	-1	1
{4}	{1,3}	{1,4}	{1,2,4}	1	1	-1	1
{4}	{1,3}	{2,3}	{3,4}	2	1	1	-1
{4}	{1,3}	{3,4}	{2,3,4}	1	1	-1	1
{4}	{1,4}	{2,4}	{3,4}	-2	1	1	1
{4}	{2,3}	{2,4}	{1,2,4}	1	1	-1	1
{4}	{2,3}	{3,4}	{1,3,4}	1	1	-1	1
{4}	{1,2,4}	{1,3,4}	{2,3,4}	- 1/2	1/2	1/2	1/2
{1,2}	{1,3}	{2,3}	{1,2,4}	- 1/2	1/2	1/2	1
{1,2}	{1,3}	{2,3}	{1,3,4}	1/2	- 1/2	1/2	1
{1,2}	{1,3}	{2,3}	{2,3,4}	1/2	1/2	- 1/2	1
{1,2}	{1,3}	{1,2,3}	{2,3,4}	1	1	-1	1
{1,2}	{1,4}	{2,4}	{1,2,3}	- 1/2	1/2	1/2	1
{1,2}	{1,4}	{2,4}	{1,3,4}	1/2	- 1/2	1/2	1
{1,2}	{1,4}	{2,4}	{2,3,4}	1/2	1/2	- 1/2	1
{1,2}	{1,4}	{1,2,4}	{2,3,4}	1	1	-1	1
{1,2}	{2,3}	{1,2,3}	{1,3,4}	1	1	-1	1
{1,2}	{2,4}	{1,2,4}	{1,3,4}	1	1	-1	1
{1,3}	{1,4}	{3,4}	{1,2,3}	- 1/2	1/2	1/2	1
{1,3}	{1,4}	{3,4}	{1,2,4}	1/2	- 1/2	1/2	1
{1,3}	{1,4}	{3,4}	{2,3,4}	1/2	1/2	- 1/2	1
{1,3}	{1,4}	{1,3,4}	{2,3,4}	1	1	-1	1
{1,3}	{2,3}	{1,2,3}	{1,2,4}	1	1	-1	1
{1,3}	{3,4}	{1,2,4}	{1,3,4}	1	1	1	-1
{1,4}	{2,4}	{1,2,3}	{1,2,4}	1	1	1	-1
{1,4}	{3,4}	{1,2,3}	{1,3,4}	1	1	1	-1
{2,3}	{2,4}	{3,4}	{1,2,3}	- 1/2	1/2	1/2	1
{2,3}	{2,4}	{3,4}	{1,2,4}	1/2	- 1/2	1/2	1

Minimal negative balanced (continued)

Collections				Weights			
{2,3}	{2,4}	{3,4}	{1,3,4}	1/2	1/2	- 1/2	1
{2,3}	{2,4}	{1,3,4}	{2,3,4}	1	1	1	-1
{2,3}	{3,4}	{1,2,4}	{2,3,4}	1	1	1	-1
{2,4}	{3,4}	{1,2,3}	{2,3,4}	1	1	1	-1

Minimal subbalanced

Collections					Standardized weights				
{1}	{2}		{1,2}	{1,2,3,4}	1	1		-1	1
{1}	{3}		{1,3}	{1,2,3,4}	1	1		-1	1
{1}	{4}		{1,4}	{1,2,3,4}	1	1		-1	1
{1}	{2,3}		{1,2,3}	{1,2,3,4}	1	1		-1	1
{1}	{2,4}		{1,2,4}	{1,2,3,4}	1	1		-1	1
{1}	{3,4}		{1,3,4}	{1,2,3,4}	1	1		-1	1
{2}	{3}		{2,3}	{1,2,3,4}	1	1		-1	1
{2}	{4}		{2,4}	{1,2,3,4}	1	1		-1	1
{2}	{1,3}		{1,2,3}	{1,2,3,4}	1	1		-1	1
{2}	{1,4}		{1,2,4}	{1,2,3,4}	1	1		-1	1
{2}	{3,4}		{2,3,4}	{1,2,3,4}	1	1		-1	1
{3}	{4}		{3,4}	{1,2,3,4}	1	1		-1	1
{3}	{1,2}		{1,2,3}	{1,2,3,4}	1	1		-1	1
{3}	{1,4}		{1,3,4}	{1,2,3,4}	1	1		-1	1
{3}	{2,4}		{2,3,4}	{1,2,3,4}	1	1		-1	1
{4}	{1,2}		{1,2,4}	{1,2,3,4}	1	1		-1	1
{4}	{1,3}		{1,3,4}	{1,2,3,4}	1	1		-1	1
{4}	{2,3}		{2,3,4}	{1,2,3,4}	1	1		-1	1
{1}	{2}	{3}	{1,2,3}	{1,2,3,4}	1	1	1	-1	1
{1}	{2}	{4}	{1,2,4}	{1,2,3,4}	1	1	1	-1	1
{1}	{3}	{4}	{1,3,4}	{1,2,3,4}	1	1	1	-1	1
{2}	{3}	{4}	{2,3,4}	{1,2,3,4}	1	1	1	-1	1
{1,2}	{1,3}	{2,3}	{1,2,3}	{1,2,3,4}	1/2	1/2	1/2	-1	1
{1,2}	{1,4}	{2,4}	{1,2,4}	{1,2,3,4}	1/2	1/2	1/2	-1	1
{1,4}	{1,3}	{3,4}	{1,3,4}	{1,2,3,4}	1/2	1/2	1/2	-1	1
{2,4}	{3,4}	{2,3}	{2,3,4}	{1,2,3,4}	1/2	1/2	1/2	-1	1

Chapter 5

Matching situations: connecting assignment and permutation

5.1 Introduction

This chapter, which is based on Tejada et al. (2011), discusses a general framework for games derived from a non-negative, square matrix in which every entry represents the value obtained from combining the corresponding row and column. We assume that every row and every column is associated with a player, where every player is associated to at most one row and at most one column. The instances arising from this framework will be called matching situations. These matching situations provide a generalization of two models known from the literature. First, in the special case that every player is associated with exactly one row or one column only, the model boils down to the assignment problem (cf. Shapley and Shubik (1972)). The assignment problem is e.g. used to model a two-sided market of heterogeneous goods, where every player associated with a row demands one good and every player associated with a column offers one good. Second, if every player is associated with exactly one column and exactly one row, the model represents a permutation problem (cf. Tijs et al. (1984)). This model is used to model a situation where every player owns a machine and a job, and costs can be reduced by processing one's job on the machine of another player. For every matching situation with the same underlying matrix, the optimization problem is the same: which assignment of rows to columns results in the maximum total value.

We analyze matching situations from a game theoretic perspective. Assignment games, the cooperative games originating from assignment problems, have been studied extensively (see, e.g., Núñez and Rafels (2002) and Martínez-Albéniz et al.

(2011)). The relation between assignment games and permutation games was studied by Curiel and Tijs (1986) and Quint (1996). As they show, every assignment game can also be obtained from a permutation situation with a different underlying matrix. For matching games, the games obtained from matching situations, we show that the core is non-empty. In doing so we generalize the mentioned results from Curiel and Tijs (1986) and Quint (1996) to our setting.

After that, this chapter consists of two parts. First, we analyze the relation between matching situations based on the same underlying matrix, but with a different player set. Second, we provide an addition to the literature on permutation situations. We focus on the structure of matrices leading to the same core of the permutation game, and the structure of the core in a specific subclass of the class of permutation situations.

To provide more detail, in the first part we associate to every assignment situation a number of matching situations with the same underlying matrix up to a reordering of the rows and columns. Tijs et al. (1984) and Quint (1996) showed that if the associated matching situation is a permutation situation, the core of the permutation game coincides with a translation of the core of the assignment game. We generalize one part of this result to obtain that the core of an associated matching game is a subset of a translation of the core of the assignment game. For every assignment situation and any associated matching situation, we show how all extreme points of the core of the matching game can be viewed as extreme points of the core of the assignment game. In general not all extreme points of the core of the assignment game are covered in this way, but for every assignment situation there exists an associated permutation situation such that all extreme points of the core of the assignment game are covered.

We introduce the class of matching situations called Böhm-Bawerk matching situations. This is a generalization of the class of Böhm-Bawerk assignment situations (Böhm-Bawerk (1891)), that consists of those assignment situations where the goods are homogeneous, and the value of combining a row and a column is given by the value of a transaction between the player associated with the row (the buyer) and the player associated with the column (the seller). Böhm-Bawerk matching situations are interesting from a mathematical point of view, as all extreme points of the core of a Böhm-Bawerk assignment game are covered by extreme points of the core of the permutation game, for any associated permutation situation. Also, we provide an expression for the nucleolus of Böhm-Bawerk permutation games and show that the nucleolus of the permutation game can be obtained from the nucleolus of the associated Böhm-Bawerk assignment game.

In the second part attention is shifted to permutation games only. We study the structure of the set of all matrices that lead to permutations games with the same core. For assignment games, the structure of this set was studied by Martínez-Albéniz et al. (2011). Also, they show that within this set there exists a unique matrix for which no entry can be raised without changing the core. For permutation games, there can be more than one such matrix. Interestingly, we show that for small instances every such matrix leads to an exact game whereas for assignment games this is not guaranteed.

Moreover, we study a specific subclass of permutation situations called 'homogeneous alternatives' permutation situations. For 'homogeneous alternatives' permutation situations, every player considers the objects of the other players to be homogeneous but we allow for a different valuation of his own object. Hence, the value obtained by a player while combining his row with the column of any another player is independent of the column player, but this value can differ from the value obtained from matching his own row with his own column. For this class of permutation situations we again focus on the core and nucleolus. For all 'homogeneous alternatives' permutation situations we provide explicit expressions for both the core and the nucleolus.

This chapter is organized as follows. Section 2 introduces matching situations and games, and reviews known results on (the relation between) assignment games and permutation games. Also, we show that matching games have a non-empty core. In section 3 we formalize the relation between assignment situations and associated matching situations, we consider the relation between the extreme points of the core of the assignment game and the extreme points of the core of an associated permutation game. Also, we consider the class of Böhm-Bawerk matching situations. More specifically, we analyze both the extreme points of the core and the nucleolus for Böhm-Bawerk permutation games and consider the preservation of both the extreme points of the core and the nucleolus in the translation from Böhm-Bawerk assignment games to Böhm-Bawerk permutation games. In the last section, on permutation situations, we study the set of permutation situations leading to the same core of the permutation game. Also, we provide an analysis of 'homogeneous alternatives' permutation situations.

5.2 A unifying model

This section introduces matching situations and games, and reviews a number of known results regarding the core of the games originating from two special classes of matching situations. Also, we show that for every matching situation, the corresponding matching game has a non-empty core.

A *matching situation* is a triple (N^1, N^2, A). Here, N^1 and N^2 are two finite player sets and A is a non-negative, $N^1 \times N^2$ matrix. We do not require N^1 and N^2 to be of equal cardinality. We define $\mathcal{M}^+_{N^1\times N^2}$ as the set of all non-negative $N^1 \times N^2$ matrices. For $i \in N^1$ and $j \in N^2$, the entry a_{ij} of the matrix A represents the value when agents i and j are paired. We allow for the player sets N^1 and N^2 to overlap, so a player might be associated to both a row and a column of the matrix A and can be paired to himself. From a combinatorial perspective, the question is how to maximize total benefits, i.e., to find two sets $T^1 \subseteq N^1$, $T^2 \subseteq N^2$ with $|T^1| = |T^2|$ and a bijection $\mu_{T^1T^2} : T^1 \to T^2$ such that the value of the bijection, given by

$$\sum_{i^1 \in T^1} a_{i^1 \mu_{T^1T^2}(i^1)}, \tag{5.1}$$

is maximized. Alternatively, we denote $(i,j) \in \mu_{T^1T^2}$ if $j = \mu_{T^1T^2}(i)$. Shapley and Shubik (1972) shows that finding a bijection that maximizes (5.1) boils down to solving a linear program. This paper considers matching situations from a game theoretic perspective: we analyze the issue of dividing total benefits among the players.

Define $N = N^1 \cup N^2$ and take $S \subseteq N$. We define $S^1 = N^1 \cap S$ and $S^2 = N^2 \cap S$. For every $S \subseteq N$ we denote the set of feasible matchings by

$$M(S^1, S^2) = \{\mu_{T^1T^2} : T^1 \to T^2 \quad | \quad T^1 \subseteq S^1, T^2 \subseteq S^2, \mu_{T^1T^2} \text{ is a bijection}\}.$$

So, for $\mu_{T^1T^2} \in M(S^1, S^2)$ the rows of players in $S^1 \backslash T^1$ are not matched to a column and the columns of players in $S^2 \backslash T^2$ are not matched to a row. The set of optimal matchings for S^1 and S^2 is given by

$$M^*_A(S^1, S^2) = \{\mu^*_{T^1T^2} \in M(S^1, S^2) \;\; | \;\; \sum_{i\in T^1} a_{i\mu^*_{T^1T^2}(i)} \geq \sum_{i \in R^1} a_{i\mu_{R^1R^2}(i)} \text{ for every } \mu_{R^1R^2} \in M(S^1, S^2)\}.$$

Note that for every $A \in \mathcal{M}^{+}_{N^1 \times N^2}$, by non-negativity of A, there always exists a $\mu^*_{T^1T^2} \in M^*_A(S^1, S^2)$ such that $|T^1| = |T^2| = \min\{|S^1|, |S^2|\}$. In particular, if $|S^1| = |S^2|$ there exists a $\mu^*_{S^1S^2} \in M^*_A(S^1, S^2)$.

Let (N^1, N^2, A) be a matching situation. Then the associated *matching game* (N, v_A) is given by $N = N^1 \cup N^2$ and, for every $S \in 2^N \backslash \{\emptyset\}$,

$$v_A(S) = \sum_{i \in T^1} a_{i\mu^*_{T^1T^2}(i)},$$

where $\mu^*_{T^1T^2} \in M^*_A(S \cap N^1, S \cap N^2)$.

Example 5.2.1 Let $N^1 = \{1, 2\}$, $N^2 = \{2, 3\}$ and let $A \in \mathcal{M}^{+}_{N^1 \times N^2}$ be given by

$$A : \begin{array}{c c} & \begin{array}{cc} 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \end{array}$$

The optimal matching $\mu^*_{N^1N^2} \in M^*_A(N^1, N^2)$ is such that $\mu^*_{N^1N^2}(1) = 2$ and $\mu^*_{N^1N^2}(2) = 3$, resulting in $v(N) = 5$. The matching game $(N^1 \cup N^2, v_A)$ is given by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v_A(S)$	0	2	0	2	2	4	5

Since $N^1 \cap N^2 = \{2\}$, player 2 is the only player that can create a positive value on his own. ◁

Two special classes of matching situations have been analyzed in the literature before. In fact, regarding the player set of these special cases, they form the 'extreme' cases: for assignment situations N^1 and N^2 are disjoint, whereas for permutation situations N^1 coincides with N^2.

Definition 5.2.2 (cf. Shapley and Shubik (1972)) Let (N^1, N^2, A) be a matching situation. Then (N^1, N^2, A) is called an *assignment situation* if $N^1 \cap N^2 = \emptyset$.

Definition 5.2.3 (cf. Tijs et al. (1984)) Let (N^1, N^2, A) be a matching situation. Then (N^1, N^2, A) is called a *permutation situation* if $N^1 = N^2$.

If the matching situation (N^1, N^2, A) is an assignment (permutation) situation, we will refer to (N, v_A) as the *assignment (permutation) game*. First we discuss the results on the core of assignment games and permutation games. Shapley and Shubik (1972) provides an expression for the core of assignment games.

Theorem 5.2.4 (Shapley and Shubik (1972)) Let (N^1, N^2, A) be an assignment situation, and let $\mu^*_{T^1T^2} \in M^*_A(N^1, N^2)$. Then

$$C(v_A) = \left\{ (u,v) \in \mathbb{R}^{N^1}_+ \times \mathbb{R}^{N^2}_+ \;\middle|\; \begin{array}{l} u_i + v_{\mu^*_{T^1T^2}(i)} = a_{i\mu^*_{T^1T^2}(i)}, \text{ for all } i \in T^1, \\ u_i + v_j \geq a_{ij}, \text{ for all } i \in N^1, j \in N^2 \backslash \{\mu^*_{T^1T^2}(i)\}, \\ u_i = 0, \text{ for every } i \in N^1 \backslash T^1, \\ v_j = 0, \text{ for every } j \in N^2 \backslash T^2 \end{array} \right\}.$$

In particular, $C(v_A) \neq \emptyset$.

For every assignment situation, there exist two special core elements. Let (N^1, N^2, A) be an assignment situation, and let $\mu^*_{T^1,T^2} \in M^*_A(N^1, N^2)$. For $i \in N^1$, define $\bar{u}_i = v_A(N) - v_A(N \backslash \{i\})$ and for $j \in N^2$ define $\bar{v}_j = v_A(N) - v_A(N \backslash \{j\})$. Also define

$$\underline{u}_i = \begin{cases} 0 & \text{if } i \in N^1 \backslash T^1, \\ a_{i\mu^*_{T^1T^2}(i)} - \bar{v}_{\mu^*_{T^1T^2}(i)} & \text{if } i \in T^1, \end{cases}$$

for every $i \in N^1$ and

$$\underline{v}_j = \begin{cases} 0 & \text{if } j \in N^2 \backslash T^2, \\ a_{(\mu^*)^{-1}_{T^1T^2}(j)j} - \bar{u}_{(\mu^*)^{-1}_{T^1T^2}(j)} & \text{if } j \in T^2, \end{cases}$$

for every $j \in N^2$.

Theorem 5.2.5 (Leonard (1983), Demange (1982)) Let (N^1, N^2, A) be an assignment situation, and let $\mu^*_{T^1,T^2} \in M^*_A(N^1, N^2)$. Then both $(\underline{u}, \bar{v}) \in C(v_A)$ and $(\bar{u}, \underline{v}) \in C(v_A)$ while

$$C(v_A) \subseteq \{(u,v) \in \mathbb{R}^{N^1}_+ \times \mathbb{R}^{N^2}_+ \mid \underline{u} \leq u \leq \bar{u}, \underline{v} \leq v \leq \bar{v}\}.$$

For permutation games, it is known that the core is non-empty.

Theorem 5.2.6 (Tijs et al. (1984)) Let (N^1, N^2, A) be a permutation situation. Then $C(v_A) \neq \emptyset$.

In fact, as every subgame of a permutation game is again a permutation game, every permutation game is totally balanced. Tijs et al. (1984) shows that any totally balanced game with at most three players is a permutation game. The following theorem, which is a generalization of a theorem by Curiel and Tijs (1986), shows that every matching game is a permutation game defined by a different matrix from the original.

Theorem 5.2.7 Let (N^1, N^2, A) be a matching situation and let $(N^1 \cup N^2, N^1 \cup N^2, B)$ be the permutation situation where $B \in \mathcal{M}^+_{N^1\cup N^2 \times N^1 \cup N^2}$ is such that for every $i, j \in N^1 \cup N^2$

$$b_{ij} = \begin{cases} a_{ij} & \text{if } i \in N^1 \text{ and } j \in N^2 \\ 0 & \text{if } i \in N^2 \backslash N^1 \text{ or } j \in N^1 \backslash N^2. \end{cases}$$

Then $v_A(S) = v_B(S)$ for all $S \in 2^{N^1 \cup N^2}$.

Proof: Let $N = N^1 \cup N^2$ and take $S \in 2^N$.
First, we show that $v_B(S) \leq v_A(S)$. Take $\mu^*_{SS} \in M^*_B(S, S)$. Note that such a μ^*_{SS} indeed exists. Take $T^1 = \{i \in S^1 \mid \mu^*_{SS}(i) \in S^2\}$ and $T^2 = \mu^*_{SS}(T^1)$, and define $\mu_{T^1T^2} \in M(S^1, S^2)$ such that $\mu_{T^1T^2}(i) = \mu^*_{SS}(i)$ for every $i \in T^1$. Now

$$v_B(S) = \sum_{i \in S} b_{i\mu^*_{SS}(i)} = \sum_{i \in T^1} b_{i\mu^*_{SS}(i)} = \sum_{i \in T^1} a_{i\mu_{T^1T^2}(i)} \leq v_A(S)$$

where the second equality follows from $b_{i\mu^*_{SS}(i)} = 0$ for every $i \in S \backslash T^1$.
Second, we show that $v_A(S) \leq v_B(S)$. Take $\mu^*_{T^1T^2} \in M^*_A(S^1, S^2)$, and define $\mu_{SS} \in M(S, S)$ such that $\mu_{SS}(i) = \mu^*_{T^1T^2}(i)$ for every $i \in T^1$ and the players in $S \backslash T^1$ are matched arbitrarily to the players in $S \backslash T^2$. We have

$$v_A(S) = \sum_{i \in T^1} a_{i\mu^*_{T^1T^2}(i)} = \sum_{i \in T^1} b_{i\mu_{SS}(i)} \leq \sum_{i \in S} b_{i\mu_{SS}(i)} \leq v_B(S)$$

where the second equality follows by construction of B and the observation that $\mu^*_{T^1T^2}(i) \in T^2$ for every $i \in T^1$. Hence, $v_A(S) = v_B(S)$ for every $S \in 2^N$. □

The previous theorem and Theorem 5.2.6 result in the following corollary:

Corollary 5.2.8 *Let (N^1, N^2, A) be a matching situation. Then $C(v_A) \neq \emptyset$.*

5.3 Assignment versus permutation

In this section we formalize the relation between assignment situations and matching situations with the same underlying matrix up to a reordering of the rows and columns. We extend the results on this relation in two ways. First, we generalize a result by Tijs et al. (1984) and Quint (1996), and show that the core of a matching game is a subset of a translation of the core of the corresponding assignment game. Second, we take a look at the preservation of specific points of the core in the translation from an assignment situation to associated permutation situations. In general, the extreme points of the core of the permutation game form a subset of the translated extreme points of the core of the assignment game and for every assignment game there exists a permutation game based on the same data, for which the reverse statement also holds. When we consider Böhm-Bawerk assignment situations, we even show that the translated extreme points of the core of the assignment game coincide with the extreme points of the core of any associated permutation game. Also, the nucleolus is preserved in the translation from a Böhm-Bawerk assignment situation to any associated permutation situation.

5.3.1 Assignment games versus permutation games

For finite player sets N and N' such that $|N| = |N'|$, we denote $\Pi(N, N')$ for the set of bijections $\sigma : N \to N'$.

Definition 5.3.1 Let (N^1, N^2, A) be an assignment situation, and let $\bar{N}^1$ and $\bar{N}^2$ be such that $|\bar{N}^1| = |N^1|$ and $|\bar{N}^2| = |N^2|$. Let $\sigma^1 \in \Pi(\bar{N}^1, N^1)$ and $\sigma^2 \in \Pi(\bar{N}^2, N^2)$. Then the *associated matching situation* $(\bar{N}^1, \bar{N}^2, \bar{A})$ is given by $\bar{a}_{ij} = a_{\sigma^1(i)\sigma^2(j)}$ for every $i, j \in N$.

So, for an assignment situation (N^1, N^2, A), the matrix A and the matrix $\bar{A}$ underlying the associated matching situation $(\bar{N}^1, \bar{N}^2, \bar{A})$ are equal as an $|N^1| \times |N^2|$ matrix up to a permutation and relabeling of the rows and columns.

Definition 5.3.2 Let $N^1 \cap N^2 = \emptyset$, let $\bar{N}^1$ and $\bar{N}^2$ be such that $|N^1| = |\bar{N}^1|$ and $|N^2| = |\bar{N}^2|$, let $\sigma^1 \in \Pi(\bar{N}^1, N^1)$, $\sigma^2 \in \Pi(\bar{N}^2, N^2)$ and let $\bar{N} = \bar{N}^1 \cup \bar{N}^2$. Then $m^{\sigma^1\sigma^2} : \mathbb{R}_+^{N^1} \times \mathbb{R}_+^{N^2} \to \mathbb{R}_+^{\bar{N}}$ is defined by

$$m_i^{\sigma^1\sigma^2}(u,v) = \begin{cases} u_{\sigma^1(i)} & \text{if } i \in \bar{N}^1 \backslash \bar{N}^2 \\ v_{\sigma^2(i)} & \text{if } i \in \bar{N}^2 \backslash \bar{N}^1 \\ u_{\sigma^1(i)} + v_{\sigma^2(i)} & \text{if } i \in \bar{N}^1 \cap \bar{N}^2 \end{cases}$$

for every $(u,v) \in \mathbb{R}_+^{N^1} \times \mathbb{R}_+^{N^2}$ and every $i \in \bar{N}$.

So, $m^{\sigma^1\sigma^2}$ maps an allocation for the assignment situation (N^1, N^2, A) into an allocation for the associated matching situation $(\bar{N}^1, \bar{N}^2, \bar{A})$. The following example illustrates the definitions above.

Example 5.3.3 Let $N^1 = \{1,2\}$, $N^2 = \{3,4\}$ and let $A \in \mathcal{M}^+_{N^1 \times N^2}$ be given by

$$A : \begin{array}{c c} & \begin{array}{cc} 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \end{array}$$

Let $\bar{N}^1 = \{1,2\}, \bar{N}^2 = \{2,3\}$, and define $\sigma^1 \in \Pi(\bar{N}^1, N^1)$, $\sigma^2 \in \Pi(\bar{N}^2, N^2)$ such that $\sigma^1(1) = 1$, $\sigma^1(2) = 2$, $\sigma^2(2) = 3$, $\sigma^2(3) = 4$. Then we have for the matching situation $(\bar{N}^1, \bar{N}^2, \bar{A})$ associated with (N^1, N^2, A):

$$\bar{A} : \begin{array}{c c} & \begin{array}{cc} 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \end{array}$$

The allocation $((0,1),(1,3)) \in C(v_A)$ for the assignment game corresponds with the allocation $m^{\sigma^1\sigma^2}((0,1),(1,3)) = (0,2,3)$ for the matching game $(\bar{N}^1, \bar{N}^2, \bar{A})$. Since $C(v_{\bar{A}}) = \text{conv}\{(0,2,3),(0,3,2),(1,2,2),(1,3,1)\}$, we have that $(0,2,3) \in C(v_{\bar{A}})$. $\triangleleft$

Tijs et al. (1984) and Quint (1996) studied the relation between the core of an assignment game and the core of an associated permutation game. For an assignment game, such an associated permutation game only exists if the underlying assignment situation is square, i.e., $|N^1| = |N^2|$. For an assignment situation (N^1, N^2, A) and an associated permutation situation $(\bar{N}^1, \bar{N}^2, \bar{A})$, the translation of the core of the assignment game $(N^1 \cup N^2, v_A)$ coincides with the core of the permutation game $(\bar{N}^1 \cup \bar{N}^2, v_{\bar{A}})$.

Theorem 5.3.4 (Tijs et al. (1984), Quint (1996)) Let (N^1, N^2, A) be a square assignment situation, and let $\bar{N}$ be such that $|\bar{N}| = |N^1|$. Let $\sigma^1 \in \Pi(\bar{N}, N^1)$ and $\sigma^2 \in \Pi(\bar{N}, N^2)$. Then for the associated permutation situation $(\bar{N}, \bar{N}, \bar{A})$, we have $m^{\sigma^1\sigma^2}(C(v_A)) = C(v_{\bar{A}})$.

The above result shows that the core is preserved in the translation from the assignment situation (N^1, N^2, A) to the permutation situation $(\bar{N}, \bar{N}, \bar{A})$. This raises the question whether specific points in the core are preserved by this translation. First, we consider the relation between the extreme points of the core of the assignment game (N, v_A) and the extreme points of the core of the associated matching game $(N, v_{\bar{A}})$. It turns out that the extreme points of the core of the matching game form a subset of the translated extreme points of the core of the assignment game. It is important to note that the theorem holds for any associated matching game, not only if the associated matching game is a permutation game.

Theorem 5.3.5 Let (N^1, N^2, A) be an assignment situation, let $\bar{N}^1$ and $\bar{N}^2$ be such that $|\bar{N}^1| = |N^1|$ and $|\bar{N}^2| = |N^2|$. Let $\sigma^1 \in \Pi(\bar{N}^1, N^1)$ and $\sigma^2 \in \Pi(\bar{N}^2, N^2)$. Then

$$\text{ext}\,(C(v_{\bar{A}})) \subseteq m^{\sigma^1\sigma^2}(\text{ext}(C(v_A))).$$

Proof: Let $y \in \text{ext}\,\{C(v_{\bar{A}})\}$ and let $P^1 = C(v_A)$, which is a polytope. Let also $S^1 = \{x \in P^1 \mid m^{\sigma^1\sigma^2}(x) = y\}$. By Theorem 5.3.4, S^1 is nonempty.
If P^1 is a singleton, the result in the theorem trivially holds. Hence, we assume that $\dim(P^1) > 0$. We show that there is a finite chain $P^1 \supset P^2 \supset .... \supset P^t$, such that P^2 is a facet of P^1, P^3 is a facet of P^2 etc., and $S^k = \{x \in P^k \mid m^{\sigma^1\sigma^2}(x) = y\} \neq \emptyset$ for all $k \in \{1, ..., t\}$ and $\dim(P^t) = 0$. This finishes the proof, as the only element of $x \in P^t$ is an extreme point of $C(v_{\bar{A}})$ and $m^{\sigma^1\sigma^2}(x) = y$.
It suffices to show that S^1 cannot be contained in the relative interior of P^1, $\text{int}(P^1)$: this implies that there must be one facet of P^1, let us say P^2, such that $S^2 := S^1 \cap P^2 = (m^{\sigma^1\sigma^2})^{-1}(y) \cap P^2 \neq \emptyset$. Since P^2 is a facet of P^1, $\text{ext}\,\{P^2\} \subset \text{ext}\,\{P^1\}$ and $\dim(P^2) < \dim(P^1)$. If $\dim(P^2) = 0$, i.e., $P^2 = \{p\}$, the proof is complete since $y = m^{\sigma^1\sigma^2}(p)$ and $p \in \text{ext}\,\{C(v_A)\}$. Otherwise, i.e., $\dim(P^2) > 0$, repeating the argument for $S^1 \nsubseteq \text{int}(P^1)$ proves that $S^2 \nsubseteq \text{int}(P^2)$ which implies that there exists a facet P^3 of P^2 such that $S^3 = S^2 \cap P^3 = (m^{\sigma^1\sigma^2})^{-1}(y) \cap P^3 \neq \emptyset$. Iterating these arguments a finite number of times, there must be a facet $P^t = \{p\}$ of P^{t-1}, for some $t > 2$, such that $S^t := S^{t-1} \cap P^t = (m^{\sigma^1\sigma^2})^{-1}(y) \cap P^t \neq \emptyset$ and $\dim(P^t) = 0$. That

is, $y = m^{\sigma^1\sigma^2}(p)$ and $p \in \text{ext}\,\{C(v_A)\}$, since $\text{ext}(P^t) \subset \text{ext}(P^{t-1}) \subset ... \subset \text{ext}(P^1) = \text{ext}(C(v_A))$.
So, we show via contradiction that $S^1 \nsubseteq \text{int}(P^1)$. Suppose $S^1 \subseteq \text{int}(P^1)$ and let p^1 be any extreme point of P^1. On the one hand, since $m^{\sigma^1\sigma^2}$ is a continuous function, S^1 is a compact set. Hence, there is $x^1 \in S^1$ that is the closest point to p^1 within the set S^1 with respect to the Euclidean distance, i.e., $d(p^1, x) \geq d(p^1, x^1) = \delta > 0$ for any $x \in S^1$. On the other hand, since $x^1 \in \text{int}(P^1)$, there is $\varepsilon > 0$ such that $B_\varepsilon(x^1) \subseteq \text{int}(P^1)$.
Let $x' \in B_\delta(p^1) \cap B_\varepsilon(x^1)$ and $x'' \in P^1$ such that $x^1 = \frac{1}{2}x' + \frac{1}{2}x''$. Therefore, by linearity of $m^{\sigma^1\sigma^2}$

$$y \;=\; m^{\sigma^1\sigma^2}(x^1) = \frac{1}{2}m^{\sigma^1\sigma^2}(x') + \frac{1}{2}m^{\sigma^1\sigma^2}(x''),$$

where, by (5.3.4), $m^{\sigma^1\sigma^2}(x') \in C(v_{\bar{A}})$ and $m^{\sigma^1\sigma^2}(x'') \in C(v_{\bar{A}})$. Since $y \in \text{ext}\,\{C(v_{\bar{A}})\}$, this implies that $m^{\sigma^1\sigma^2}(x') = m^{\sigma^1\sigma^2}(x'') = y$ and hence $x' \in S^1$ which contradicts $x' \in B_\delta(p^1)$. Therefore our supposition was incorrect, i.e., it must be the case that $S^1 \nsubseteq \text{int}(P^1)$. □

It is important to point out that the reverse inclusion does not hold in general.

Example 5.3.6 Let (N^1, N^2, A) be an assignment situation, where

$$A : \begin{array}{c} \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \end{array},$$

Total profit is optimized by $\mu^*_{N^1N^2} \in M^*_A(N^1, N^2)$ such that $\mu^*_{N^1N^2}(1) = 4$ and $\mu^*_{N^1N^2}(2) = 3$.

The assignment game $(N^1 \cup N^2, v_A)$ is given by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$
$v_A(S)$	0	0	0	0	0	1	2	2	0	0

S	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
$v_A(S)$	2	2	2	2	4

E.g., for coalition $\{1, 2, 4\}$ the optimal mapping $\mu^*_{\{1\}\{4\}} \in M^*_A(\{1, 2\}, \{4\})$ is such that $\mu^*_{\{1\}\{4\}}(1) = 4$ and player 2 is not matched. Using Theorem 5.2.4, we have $C(v_A) = \text{conv}\{(0, 0, 2, 2), (0, 1, 1, 2), (1, 2, 0, 1), (2, 0, 2, 0), (2, 2, 0, 0)\}$.

Let $\bar{N} = \{1,2\}$, and let $\sigma^1 \in \Pi(\bar{N}^1, N^1)$ and $\sigma^2 \in \Pi(\bar{N}^2, N^2)$ be such that $\sigma^1(1) = 1$, $\sigma^1(2) = 2$, $\sigma^2(3) = 1$, $\sigma^2(4) = 2$. This results in the permutation situation $(\bar{N}, \bar{N}, \bar{A})$, with

$$A : \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \end{array}.$$

For the associated permutation game $(\bar{N}, v_{\bar{A}})$ we obtain:

S	$\{1\}$	$\{2\}$	$\{1,2\}$
$v_{\bar{A}}(S)$	1	0	4

and the core is given by $C(v_{\bar{A}}) = \text{conv}\{(1,3),(4,0)\}$. Since $(0,0,2,2) \in \text{ext}(C(v_A))$ but $m((0,0,2,2)) = (2,2) \notin \text{ext}(C(v_{\bar{A}}))$, we can conclude that $\text{ext}\,\{C(v_{\bar{A}})\} \not\supseteq m^{\sigma^1\sigma^2}(\text{ext}(C(v_A)))$. ◁

Nevertheless, with every square assignment situation we can associate a permutation situation such that the reverse inclusion holds.

Theorem 5.3.7 Let (N^1, N^2, A) be an assignment situation such that $|N^1| = |N^2|$, and let $\bar{N}$ be such that $|\bar{N}| = |N^1|$. Then there exist $\sigma^1 \in \Pi(\bar{N}, N^1)$ and $\sigma^2 \in \Pi(\bar{N}, N^2)$ such that for the associated permutation situation $(\bar{N}, \bar{N}, \bar{A})$ it holds that

$$m^{\sigma^1\sigma^2}(\text{ext}(C(v_A))) = \text{ext}(C(v_{\bar{A}})).$$

Proof: Let $\mu^*_{N^1N^2} \in M^*_A(N^1, N^2)$. Let $\sigma^1 \in \Pi(\bar{N}, N^1)$, and let $\sigma^2 \in \Pi(\bar{N}, N^2)$ be such that $\sigma^2(i) = \mu^*_{N^1N^2}(\sigma^1(i))$ for every $i \in N$. By Theorem 5.3.5 we have $\text{ext}(C(v_{\bar{A}})) \subseteq m^{\sigma^1\sigma^2}(\text{ext}(C(v_A)))$.
It remains to show that $m^{\sigma^1\sigma^2}(\text{ext}(C(v_A))) \subseteq \text{ext}(C(v_{\bar{A}}))$. It is readily checked that $v_{\bar{A}}$ is an additive game and $\text{ext}(C(v_{\bar{A}})) = C(v_{\bar{A}}) = (\bar{a}_{ii})_{i \in N}$. Let $(u,v) \in \text{ext}(C(v_A))$. By Theorem 5.2.4 we have $u_i + v_j = a_{ij}$ for all $(i,j) \in \mu^*_{N^1N^2}$. Hence, for every $i \in N$, $m_i^{\sigma^1\sigma^2}(u,v) = u_{\sigma^1(i)} + v_{\sigma^2(i)} = a_{\sigma^1(i)\sigma^2(i)} = \bar{a}_{ii}$. Therefore, $m^{\sigma^1\sigma^2}(u,v) \in \text{ext}(C(v_{\bar{A}}))$ so $m^{\sigma^1\sigma^2}(\text{ext}(C(v_A))) \subseteq \text{ext}(C(v_{\bar{A}}))$. □

5.3.2 Böhm-Bawerk matching

Assignment situations can be used to model a two-sided market of heterogeneous goods. The Böhm-Bawerk horse market situation (Böhm-Bawerk (1891)) is a specific assignment situation where the goods traded for money are homogeneous. In a Böhm-Bawerk assignment situation every player is characterized by a single non-negative number: each buyer $i \in N^1$ values a unit of the good at $w_i \geq 0$, whereas each seller $j \in N^2$ values her own good at $c_j \geq 0$. The profit that can be reached through a transaction between buyer i and seller j is $\max\{0, w_i - c_j\}$. The following definition generalizes Böhm-Bawerk assignment situations.

Definition 5.3.8 Let N^1, N^2 be two player sets, and let $w \in \mathbb{R}_+^{N^1}$ and $c \in \mathbb{R}_+^{N^2}$. Then the *Böhm-Bawerk matching situation* $(N^1, N^2, A^{w,c})$ is given by $a_{ij}^{w,c} = \max\{0, w_i - c_j\}$ for every $i \in N^1$ and $j \in N^2$.

If $(N^1, N^2, A^{w,c})$ is an assignment situation, we obtain the original model by Böhm-Bawerk (1891).

We expand the Böhm-Bawerk framework, but the interpretation of the Böhm-Bawerk assignment situation cannot be expanded to the Böhm-Bawerk matching situation, as it seems unnatural to have a player simultaneously being a seller with a low valuation of the good and a buyer with a high valuation of the good. The following is however interesting from a mathematical point of view, as for this class of matching situations both the extreme points of the core and the nucleolus are preserved in the translation from an assignment situation to the associated permutation situation. Furthermore, with a different interpretation - one can think of situations where processing the job of player $i \in N^1$ on any machine results in a revenue of w_i, and c_j represents the cost of processing a job on the machine of player $j \in N^2$ - the results remain relevant.

For every Böhm-Bawerk matching situation, we can characterize an optimal order.

Theorem 5.3.9 (Böhm-Bawerk (1891)) Let $(N^1, N^2, A^{w,c})$ be a Böhm-Bawerk matching situation. Then there exists a matching $\mu^*_{T^1T^2} \in M^*_{A^{w,c}}(N^1, N^2)$ such that $w_i \geq w_j$ for every $i \in T^1$ and every $j \in N^1 \backslash T^1$, $c_i \leq c_j$ for every $i \in T^2$ and every $j \in N^2 \backslash T^2$, and, for every $i, j \in T^1$, $c_{\mu^*_{T^1T^2}(i)} \leq c_{\mu^*_{T^1T^2}(j)}$ if $w_i > w_j$.

So, for such an optimal matching $\mu^*_{T^1T^2} \in M^*_{A^{w,c}}(N^1, N^2)$, the buyer with the highest valuation is paired to the seller with the lowest valuation, the buyer with the second-highest valuation is paired to the seller with the second-lowest valuation etc.

For every Böhm-Bawerk matching situation $(N^1, N^2, A^{w,c})$, we define the following parameter

$$r^{w,c} = \left| \left\{ i \in N^1 \mid a^{w,c}_{i\mu^*_{T^1T^2}(i)} > 0 \right\} \right|,$$

where $\mu^*_{T^1T^2} \in M^*_{A^{w,c}}(N^1, N^2)$ is such that it satisfies the conditions of Theorem 5.3.9. Given the Böhm-Bawerk permutation game associated to $A^{w,c}$, $r^{w,c}$ is the maximum number of matchings that can be carried out simultaneously, giving a strictly positive benefit. To avoid the null game we assume $r^{w,c} > 0$.

For the vectors $w \in \mathbb{R}^{N^1}$ and $c \in \mathbb{R}^{N^2}$, we define $\tilde{w} \in \mathbb{R}^{N^1}$ as the vector that orders the elements of w such that $\tilde{w}_1 \geq \tilde{w}_2 \geq ... \geq \tilde{w}_n$ and $\tilde{c} \in \mathbb{R}^{N^2}$ as the vector that orders the elements of c such that $\tilde{c}_1 \leq \tilde{c}_2 \leq ... \leq \tilde{c}_n$. Hence, $\tilde{w}_k$ denotes the kth largest component of w, and $\tilde{c}_k$ denotes the kth smallest component of c. For notational convenience, we take $\tilde{w}_k = -\infty$ if $k > n$ and $\tilde{c}_k = \infty$ if $k > n$.

For the Böhm-Bawerk matching situation $(N^1, N^2, A^{w,c})$, we say that an agent $i \in N^1$ is *buyer-active* if $w_i \geq \tilde{w}_{r^{w,c}}$. We denote $H^B \in 2^{N^1 \cup N^2}$ for the set of buyer-active agents, and denote $H^B_S = H^B \cap S$ for every $S \in 2^{N^1 \cup N^2}$. We call an agent $i \in N^2$ *seller-active* if $c_i \leq \tilde{c}_{r^{w,c}}$. We denote $H^S \in 2^{N^1 \cup N^2}$ for the set of seller-active agents, and denote $H^S_S = H^S \cap S$ for every $S \in 2^{N^1 \cup N^2}$. A player $i \in N^1 \cap N^2$ is called *active* if he is both buyer-active and seller-active, and is called *inactive* if he is neither buyer-active nor seller-active. As the player $i \in N$ such that $w_i = \tilde{w}_{r^{w,c}}$ (as well as the player $i \in N$ such that $c_i = \tilde{c}_{r^{w,c}}$) need not be unique, the number of active buyers can differ from the number of active sellers.

Define for the Böhm-Bawerk assignment situation $(N^1, N^2, A^{w,c})$, $\underline{p}, \bar{p} \in \mathbb{R}_+$ such that

$$\begin{aligned} \underline{p} &= \max\left\{\tilde{w}_{r^{w,c}+1}, \tilde{c}_{r^{w,c}}\right\}, \\ \bar{p} &= \min\left\{\tilde{w}_{r^{w,c}}, \tilde{c}_{r^{w,c}+1}\right\}. \end{aligned}$$

The values for $\underline{p}$ and $\bar{p}$ are such that the number of transactions such that all sellers and all buyers involved in a transaction achieve a non-negative value, is maximized. Given an optimal matching $\mu^*_{T^1T^2} \in M^*_{A^{w,c}}(N^1, N^2)$, a price $p \in [\underline{p}, \bar{p}]$ and player $i \in T^1$, if the value $w_i - c_{\mu^*_{T^1T^2}(i)}$ from a transaction between the matched players i

and $\mu^*_{T^1T^2}(i)$ is non-negative, then both the value $w_i - p$ that the buyer i achieves from buying the good and the value $p - c_{\mu^*_{T^1T^2}(i)}$ that the seller $\mu^*_{T^1T^2}(i)$ achieves from selling the good are non-negative.

For Böhm-Bawerk assignment games, the reverse inclusion of Theorem 5.2.5 also holds: the core is a line segment defined by $(\underline{u}, \bar{v})$ and $(\bar{u}, \underline{v})$. Moreover, we can provide expressions for $\underline{u}, \bar{u}, \underline{v}$ and $\bar{v}$ in terms of w and c.

Theorem 5.3.10 (Shapley and Shubik (1972), Moulin (1996)) Let $(N^1, N^2, A^{w,c})$ be a Böhm-Bawerk assignment situation. Then $C(v_{A^{w,c}}) = \text{conv}\{(\underline{u}, \bar{v}), (\bar{u}, \underline{v})\}$ with, for every $i \in N^1$,

$$\underline{u}_i = \begin{cases} w_i - \bar{p} & \text{if } i \in H^B \\ 0 & \text{if } i \in N^1 \backslash H^B \end{cases} \text{ and } \bar{u}_i = \begin{cases} w_i - \underline{p} & \text{if } i \in H^B \\ 0 & \text{if } i \in N^1 \backslash H^B. \end{cases}$$

And, for every $i \in N^2$,

$$\underline{v}_i = \begin{cases} \underline{p} - c_i & \text{if } i \in H^S \\ 0 & \text{if } i \in N^2 \backslash H^S \end{cases} \text{ and } \bar{v}_i = \begin{cases} \bar{p} - c_i & \text{if } i \in H^S \\ & 0 \text{ if } i \in N^2 \backslash H^S. \end{cases}$$

Using Theorem 5.3.4 and Theorem 5.3.10, it is straightforward to express the core of a Böhm-Bawerk permutation game in terms of the valuations of the players.

Theorem 5.3.11 Let $(N, N, A^{w,c})$ be a Böhm-Bawerk permutation game. Then $C(v_{A^{w,c}}) = \{y(p) \mid \underline{p} \leq p \leq \bar{p}\}$ where

$$y(p)_i = \begin{cases} w_i - c_i & \text{if } i \in H^B \cap H^S \\ w_i - p & \text{if } i \in H^B \backslash H^S \\ p - c_i & \text{if } i \in H^S \backslash H^B \\ 0 & \text{if } i \in N \backslash (H^B \cup H^S), \end{cases}$$

for every $p \in [\underline{p}, \bar{p}]$ and every $i \in N$.

Proof: The expression follows directly from Theorem 5.3.4 and Theorem 5.3.10. □

Observe that an active agent does not benefit from cooperation, since he receives his stand alone value at any core allocation, but only agents that are either buyer-active or seller-active (and not both) do. As it should be expected the interests of all agents that are only buyer-active are totally aligned and opposed to the interests of all agents that are only seller-active, and vice versa.

Now, we turn to the preservation of distinct points of the core. First of all, we consider the extreme points of the core.

Theorem 5.3.12 Let $(N^1, N^2, A^{w,c})$ be a square Böhm-Bawerk assignment situation, let $\bar{N}$ be such that $|\bar{N}| = |N^1|$. Then for any $\sigma^1 \in \Pi(\bar{N}, N^1)$ and $\sigma^2 \in \Pi(\bar{N}, N^2)$, it holds for permutation situation $(\bar{N}, \bar{N}, \overline{A^{w,c}})$ that $\text{ext}\,\{C(v_{\overline{A^{w,c}}})\} = m^{\sigma^1\sigma^2}(\text{ext}\,\{C(v_{A^{w,c}})\})$.

Proof: By Theorem 5.3.10, we have $\text{ext}(C(v_{A^{w,c}}) = \{(\underline{u}, \bar{v}), (\bar{u}, \underline{v})\}$. It is readily checked that $m^{\sigma^1\sigma^2}(\underline{u}, \bar{v}) = y(\bar{p})$ and $m^{\sigma^1\sigma^2}(\bar{u}, \underline{v}) = y(\underline{p})$, so by Theorem 5.3.11 the statement follows. □

Note the discrepancy with Theorem 5.3.7. For a general square assignment situation, there exist an associated permutation situation such that the extreme points of the core are preserved in the translation from the core of the assignment game to the core of the permutation game. For the special case of Böhm-Bawerk assignment situations however, this holds for every associated permutation situation.

Now, we turn to the nucleolus. For Böhm-Bawerk assignment games Núñez and Rafels (2005) show that the nucleolus coincides with the barycenter of the core.

Theorem 5.3.13 (Núñez and Rafels (2005)) Let $(N^1, N^2, A^{w,c})$ be a Böhm-Bawerk assignment situation. Then

$$\eta(N, v_{A^{w,c}}) = \frac{1}{2}(\bar{u}, \underline{v}) + \frac{1}{2}(\underline{u}, \bar{v}).$$

We provide an explicit expression for the nucleolus of a Böhm-Bawerk permutation game.

Theorem 5.3.14 Let $(N, N, A^{w,c})$ be a Böhm-Bawerk permutation situation. Then

$$\eta(N, v_{A^{w,c}}) = y\left(\frac{1}{2}\underline{p} + \frac{1}{2}\bar{p}\right).$$

Proof: First, if $\tilde{w}_{r^{w,c}} = \tilde{w}_{r^{w,c}+1}$, $\bar{p} = \min\{\tilde{w}_{r^{w,c}+1}, \tilde{c}_{r^{w,c}+1}\} = \tilde{w}_{r^{w,c}+1}$ and $\underline{p} = \max\{\tilde{w}_{r^{w,c}}, \tilde{c}_{r^{w,c}}\} = \tilde{w}_{r^{w,c}}$. Hence, $\bar{p} = \underline{p}$. Therefore, the core $C(v_{A^{w,c}})$ is a singleton and the statement follows. If $\tilde{c}_{r^{w,c}} = \tilde{c}_{r^{w,c}+1}$, a similar argument shows that $C(v_{A^{w,c}})$ is a singleton. So, we assume $\tilde{w}_{r^{w,c}} > \tilde{w}_{r^{w,c}+1}$ and $\tilde{c}_{r^{w,c}} < \tilde{c}_{r^{w,c}+1}$. This means that $|H^S| = |H^B|$. If $H^S = H^B$, then $C(v_{A^{w,c}})$ is a singleton and the statement follows. So, we assume $|H^S| = |H^B|$ and $H^S \backslash H^B \neq \emptyset$ (and therefore $H^B \backslash H^S \neq \emptyset$).

The structure of the proof is as follows: first, we show that for every coalition with the same number of active sellers and active buyers the excess is 0 for every core element. Then, we demonstrate that there are two special coalitions such that for every core element the highest excess that is not constant across all core elements, is attained by one of these two coalitions. hence, the excess vector is lexicographically minimized, if the maximum of the excess of these two coalitions is minimized.

For every $S \in 2^N$, define $\sigma_S^B \in \Pi(S)$ and $\sigma_S^S \in \Pi(S)$ such that $w_{\sigma_S^B(k)} \geq w_{\sigma_S^B(k+1)}$ and $c_{\sigma_S^S(k)} \leq c_{\sigma_S^S(k+1)}$ for every $k \in \{1, ..., |S|-1\}$. Note that for every $S \in 2^N$, there exists a $\mu^*_{SS} \in M^*_{A^{w,c}}(S,S)$ such that $\mu^*_{SS}(\sigma_S^B(k)) = \sigma_S^S(k)$ for every $k \in \{1, ..., |S|\}$.

Take $p \in [\underline{p}, \bar{p}]$. First, we show that $E(S, y(p)) = 0$ if $|H_S^S| = |H_S^B|$. So, let $S \in 2^N$ be such that $|H_S^S| = |H_S^B|$. We have

$$\begin{aligned}
E(S,y(p)) \quad + \quad & v(S) - \sum_{i \in S} y_i(p) \\
= \quad & \sum_{t=1}^{|H_S^S|} (w_{\sigma_S^B(t)} - c_{\sigma_S^S(t)}) - \sum_{i \in H_S^S \cap H_S^B} (w_i - c_i) \\
& - \sum_{i \in H_S^S \backslash H_S^B} (p - c_i) - \sum_{i \in H_S^B \backslash H_S^S} (w_i - p) \\
= \quad & 0.
\end{aligned}$$

So, for every coalition $S \in 2^N$ with $|H_S^S| = |H_S^B|$ the excess $E(S, y(p))$ is independent of p.

Now, we provide upper bounds for the excess of coalitions $S \in 2^N$ such that $|H_S^B| \neq |H_S^S|$ and show that these upper bounds are attained by some coalition.

- $|H_S^B| > |H_S^S|$. We obtain the following expression for the excess $E(S, y(p))$:

$$\begin{aligned}
E(S,y(p)) &= v(S)-\sum_{i\in S}y_i(p)\\
&= \sum_{t=1}^{|H_S^S|}(w_{\sigma_S^B(t)}-c_{\sigma_S^S(t)})+\sum_{t=|H_S^S|+1}^{|H_S^B|}\max\{w_{\sigma_S^B(t)}-c_{\sigma_S^S(t)},0\}\\
&\quad -\sum_{i\in H_S^S\cap H_S^B}(w_i-c_i)-\sum_{i\in H_S^S\setminus H_S^B}(p-c_i)-\sum_{i\in H_S^B\setminus H_S^S}(w_i-p)\\
&= \sum_{t=1}^{|H_S^S|}(w_{\sigma_S^B(t)}-c_{\sigma_S^S(t)})+\sum_{t=|H_S^S|+1}^{|H_S^B|}\max\{w_{\sigma_S^B(t)}-c_{\sigma_S^S(t)},0\}\\
&\quad -\sum_{i\in H_S^S}(p-c_i)-\sum_{i\in H_S^B}(w_i-p)\\
&= \sum_{t=|H_S^S|+1}^{|H_S^B|}(\max\{w_{\sigma_S^B(t)}-c_{\sigma_S^S(t)},0\}-(w_{\sigma_S^B(t)}-p))\\
&= \sum_{t=|H_S^S|+1}^{|H_S^B|}(\max\{p-c_{\sigma_S^S(t)},p-w_{\sigma_S^B(t)}\})\\
&\leq \max\{p-\tilde{c}_{r^{w,c}+1},p-\tilde{w}_{r^{w,c}}\},
\end{aligned}$$

where the fourth equality follows from the fact that $\sigma_S^B(t)\in H_S^B$ and $\sigma_S^S(t)\in H_S^S$ for every $t\in\{1,...,|H_S^S|\}$. The inequality follows from the observation that, for every $t\in\{|H_S^S|+1,...,|H_S^B|\}$, $\max\{p-c_{\sigma_S^S(t)},p-w_{\sigma_S^B(t)}\}\leq\max\{p-\tilde{c}_{r^{w,c}+1},p-\tilde{w}_{r^{w,c}}\}\leq 0$ as $\tilde{c}_{r^{w,c}+1}\leq c_{\sigma_S^S(t)}$, $\tilde{w}_{r^{w,c}}\leq w_{\sigma_S^B(t)}$ and $p\leq\bar{p}=\min\{\tilde{w}_{r^{w,c}},\tilde{c}_{r^{w,c}+1}\}$.

We show that there actually exists a coalition $S\in 2^N$ such that $E(S,y(p))=\max\{p-\tilde{c}_{r^{w,c}+1},p-\tilde{w}_{r^{w,c}}\}$. Take $i\in N$ such that $w_i=\tilde{w}_{r^{w,c}}$ and $j\in N$ such that $c_j=\tilde{c}_{r^{w,c}+1}$. Obviously $i\in H^B$ and $j\notin H^S$. If $i=j$, then clearly $E(\{i\},y(p))=\max\{\tilde{w}_{r^{w,c}}-\tilde{c}_{r^{w,c}+1},0\}-(\tilde{w}_{r^{w,c}}-p)=\max\{p-\tilde{c}_{r^{w,c}+1},p-\tilde{w}_{r^{w,c}}\}$. If $i\neq j$, we distinguish the following cases:

(A) $i\in H^S$ and $j\in H^B$. Consider $S=\{i,j\}$. We have $\sigma_S^B(1)=j$, $\sigma_S^B(2)=i$, $\sigma_S^S(1)=i$, $\sigma_S^S(2)=j$, so

$$\begin{aligned} E(S, y(p)) &= (\tilde{w}_{r^{w,c}} - \tilde{c}_{r^{w,c}+1}) - \max\{w_j - c_i, 0\} \\ &\quad -(\tilde{w}_{r^{w,c}} - c_i) - (w_j - p) \\ &= \max\{p - \tilde{c}_{r^{w,c}+1}, p - \tilde{w}_{r^{w,c}}\}. \end{aligned}$$

(B) $i \in H^S$ and $j \notin H^B$. Let $h \in H^B \backslash H^S$, and take $S = \{h, i, j\}$. Now $\sigma_S^B(1) = h$, $\sigma_S^B(2) = i$, $\sigma_S^B(3) = j$, $\sigma_S^S(1) = i$, $\sigma_S^S(2) = j$, $\sigma_S^S(3) = h$, so

$$\begin{aligned} E(S, y(p)) &= (w_h - c_i) + \max\{\tilde{w}_{r^{w,c}} - \tilde{c}_{r^{w,c}+1}, 0\} \\ &\quad -(w_h - p) - (\tilde{w}_{r^{w,c}} - c_i) \\ &= \max\{p - \tilde{c}_{r^{w,c}+1}, p - \tilde{w}_{r^{w,c}}\}. \end{aligned}$$

(C) $i \notin H^S$ and $j \in H^B$. Similar to (B), let $h \in H^S \backslash H^B$, and take $S = \{h, i, j\}$. Now $\sigma_S^B(1) = j$, $\sigma_S^B(2) = i$, $\sigma_S^B(3) = h$, $\sigma_S^S(1) = h$, $\sigma_S^S(2) = i$, $\sigma_S^S(3) = j$, so

$$\begin{aligned} E(S, y(p)) &= (w_j - c_h) + \max\{\tilde{w}_{r^{w,c}} - \tilde{c}_{r^{w,c}+1}, 0\} \\ &\quad -(\tilde{w}_{r^{w,c}} - p) - (w_j - p) - (p - c_h) \\ &= \max\{p - \tilde{c}_{r^{w,c}+1}, p - \tilde{w}_{r^{w,c}}\}. \end{aligned}$$

(D) $i \notin H^S$ and $j \notin H^B$. Consider $S = \{i, j\}$. We have $\sigma_S^B(1) = i$, $\sigma_S^B(2) = j$, $\sigma_S^S(1) = j$, $\sigma_S^S(2) = i$, so

$$\begin{aligned} E(S, y(p)) &= \max\{\tilde{w}_{r^{w,c}} - \tilde{c}_{r^{w,c}+1}, 0\} - (\tilde{w}_{r^{w,c}} - p) \\ &= \max\{p - \tilde{c}_{r^{w,c}+1}, p - \tilde{w}_{r^{w,c}}\}. \end{aligned}$$

So, for every coalition $S \in 2^N$ such that $|H_S^B| > |H_S^S|$,

$$E(S, y(p)) \leq \max\{p - \tilde{c}_{r^{w,c}+1}, p - \tilde{w}_{r^{w,c}}\},$$

with equality for at least one such S.

- $|H_S^S| > |H_S^B|$. We obtain the following expression for the excess $E(S, y(p))$:

$$
\begin{aligned}
E(S, y(p)) \quad &+ \quad v(S) - \sum_{i \in S} y_i(p) \\
&= \quad \sum_{t=1}^{|H_S^B|} (w_{\sigma_S^B(t)} - c_{\sigma_S^S(t)}) + \sum_{t=|H_S^B|+1}^{|H_S^S|} \max\{w_{\sigma_S^B(t)} - c_{\sigma_S^S(t)}, 0\} \\
&\quad\; - \sum_{i \in H_S^S \cap H_S^B} (w_i - c_i) - \sum_{i \in H_S^S \setminus H_S^B} (p - c_i) - \sum_{i \in H_S^B \setminus H_S^S} (w_i - p) \\
&= \quad \sum_{t=1}^{|H_S^B|} (w_{\sigma_S^B(t)} - c_{\sigma_S^S(t)}) + \sum_{t=|H_S^B|+1}^{|H_S^S|} \max\{w_{\sigma_S^B(t)} - c_{\sigma_S^S(t)}, 0\} \\
&\quad\; - \sum_{i \in H_S^S} (p - c_i) - \sum_{i \in H_S^B} (w_i - p) \\
&= \quad \sum_{t=|H_S^B|+1}^{|H_S^S|} (\max\{w_{\sigma_S^B(t)} - c_{\sigma_S^S(t)}, 0\} - (p - c_{\sigma_S^B(t)})) \\
&= \quad \sum_{t=|H_S^B|+1}^{|H_S^S|} (\max\{c_{\sigma_S^S(t)} - p, w_{\sigma_S^B(t)} - p\}) \\
&\leq \quad \max\{\tilde{w}_{r^{w,c}+1} - p, \tilde{c}_{r^{w,c}} - p\},
\end{aligned}
$$

where the fourth equality follows from the fact that $\sigma_S^B(t) \in H_S^B$ and $\sigma_S^S(t) \in H_S^S$ for every $t \in \{1, ..., |H_S^B|\}$. The inequality follows from the observation that, for every $t \in \{|H_S^B| + 1, ..., |H_S^S|\}$, $\max\{w_{\sigma_S^B(t)} - p, c_{\sigma_S^S(t)} - p\} \leq \max\{\tilde{w}_{r^{w,c}+1} - p, \tilde{c}_{r^{w,c}} - p\} \leq 0$ as $\tilde{c}_{r^{w,c}} \geq c_{\sigma_S^S(t)}$, $\tilde{w}_{r^{w,c}+1} \geq w_{\sigma_S^B(t)}$ and $p \geq \underline{p} = \max\{\tilde{w}_{r^{w,c}+1}, \tilde{c}_{r^{w,c}}\}$.

We show that there actually exists a coalition $S \in 2^N$ such that $E(S, y(p)) = \max\{\tilde{w}_{r^{w,c}+1} - p, \tilde{c}_{r^{w,c}} - p\}$. Take $i \in N$ such that $w_i = \tilde{w}_{r^{w,c}+1}$ and $j \in N$ such that $c_j = \tilde{c}_{r^{w,c}}$. Obviously $i \notin H^B$ and $j \in H^S$. If $i = j$, then clearly $E(\{i\}, y(p)) = \max\{\tilde{w}_{r^{w,c}+1} - \tilde{c}_{r^{w,c}}, 0\} - (p - \tilde{c}_{r^{w,c}}) = \max\{\tilde{w}_{r^{w,c}+1} - p, \tilde{c}_{r^{w,c}} - p\}$. If $i \neq j$, we distinguish the following cases:

(A) $i \in H^S$ and $j \in H^B$. Consider $S = \{i, j\}$. We have $\sigma_S^B(1) = j$, $\sigma_S^B(2) = i$, $\sigma_S^S(1) = i$, $\sigma_S^S(2) = j$, so

$$
\begin{aligned}
E(S, y(p)) \quad &= \quad (w_j - c_i) + \max\{\tilde{w}_{r^{w,c}+1} - \tilde{c}_{r^{w,c}}, 0\} - (p - c_i) - (w_j - \tilde{c}_{r^{w,c}}) \\
&= \quad \max\{\tilde{w}_{r^{w,c}+1} - p, \tilde{c}_{r^{w,c}} - p\}.
\end{aligned}
$$

(B) $i \in H^S$ and $j \notin H^B$. Let $h \in H^B \backslash H^S$, and take $S = \{h, i, j\}$. Now $\sigma_S^B(1) = h$, $\sigma_S^B(2) = i$, $\sigma_S^B(3) = j$, $\sigma_S^S(1) = i$, $\sigma_S^S(2) = j$, $\sigma_S^S(3) = h$, so

$$\begin{aligned} E(S, y(p)) &= (w_h - c_i) + \max\{\tilde{w}_{r^{w,c}+1} - \tilde{c}_{r^{w,c}}, 0\} \\ &\quad -(w_h - p) - (p - c_i) - (p - \tilde{c}_{r^{w,c}}) \\ &= \max\{\tilde{w}_{r^{w,c}+1} - p, \tilde{c}_{r^{w,c}} - p\}. \end{aligned}$$

(C) $i \notin H^S$ and $j \in H^B$. Similar to (B), let $h \in H^S \backslash H^B$, and take $S = \{h, i, j\}$. Now $\sigma_S^B(1) = j$, $\sigma_S^B(2) = i$, $\sigma_S^B(3) = h$, $\sigma_S^S(1) = h$, $\sigma_S^S(2) = i$, $\sigma_S^S(3) = j$, so

$$\begin{aligned} E(S, y(p)) &= (w_j - c_h) + \max\{\tilde{w}_{r^{w,c}+1} - \tilde{c}_{r^{w,c}}, 0\} - (w_i - \tilde{c}_{r^{w,c}}) - (p - c_h) \\ &= \max\{\tilde{w}_{r^{w,c}+1} - p, \tilde{c}_{r^{w,c}} - p\}. \end{aligned}$$

(D) $i \notin H^S$ and $j \notin H^B$. Consider $S = \{i, j\}$. We have $\sigma_S^B(1) = i$, $\sigma_S^B(2) = j$, $\sigma_S^S(1) = j$, $\sigma_S^S(2) = i$, so

$$\begin{aligned} E(S, y(p)) &= \max\{\tilde{w}_{r^{w,c}} - \tilde{c}_{r^{w,c}+1}, 0\} - (p - \tilde{c}_{r^{w,c}+1}) \\ &= \max\{\tilde{w}_{r^{w,c}+1} - p, \tilde{c}_{r^{w,c}} - p\}. \end{aligned}$$

So, for every coalition $S \in 2^N$ such that $|H_S^S| > |H_S^B|$,

$$E(S, y(p)) \leq \max\{\tilde{w}_{r^{w,c}+1} - p, \tilde{c}_{r^{w,c}} - p\},$$

with equality for at least one such S.

Take $S^1, S^2 \in 2^N$ such that, for every $p \in [\underline{p}, \bar{p}]$, $E(S^1, y(p)) = \max\{p - \tilde{c}_{r^{w,c}+1}, p - \tilde{w}_{r^{w,c}}\}$ and $E(S^2, y(p)) = \max\{\tilde{w}_{r^{w,c}+1} - p, \tilde{c}_{r^{w,c}} - p\}$. Irrespective of the choice of $p \in [\underline{p}, \bar{p}]$,

$$\max_{S \in 2^N \text{ s.t. } |H_S^B| \neq |H_S^S|} \{E(S, y(p))\} = \max\{E(S^1, y(p)), E(S^2, y(p))\}.$$

Since

$$\begin{aligned} E(S^1, y(p)) + E(S^2, y(p)) &= \max\{p - \tilde{c}_{r^{w,c}+1}, p - \tilde{w}_{r^{w,c}}\} \\ &\quad + \max\{\tilde{w}_{r^{w,c}+1} - p, \tilde{c}_{r^{w,c}} - p\} \\ &= \max\{\tilde{w}_{r^{w,c}+1}, \tilde{c}_{r^{w,c}}\} - \min\{\tilde{c}_{r^{w,c}+1}, \tilde{w}_{r^{w,c}}\} \\ &= \underline{p} - \bar{p}, \end{aligned}$$

is constant, the excess vector of $y(p)$ is lexicographically minimized if $E(S^1, y(p)) = E(S^2, y(p))$. Hence, for the nucleolus $\eta(v_{A^{w,c}})$ it must hold that

$$E(S^1, \eta(v_{A^{w,c}})) = E(S^2, \eta(v_{A^{w,c}})) = \frac{1}{2}(\underline{p} - \bar{p}).$$

So, we obtain

$$\begin{aligned} p &= E(S^1, \eta(v_{A^{w,c}})) - \max\{-\tilde{c}_{r^{w,c}+1}, -\tilde{w}_{r^{w,c}}\} \\ &= \frac{1}{2}(\underline{p} - \bar{p}) + \min\{\tilde{c}_{r^{w,c}+1}, \tilde{w}_{r^{w,c}}\} \\ &= \frac{1}{2}(\max\{\tilde{w}_{r^{w,c}+1}, \tilde{c}_{r^{w,c}}\} - \min\{\tilde{c}_{r^{w,c}+1}, \tilde{w}_{r^{w,c}}\}) + \min\{\tilde{c}_{r^{w,c}+1}, \tilde{w}_{r^{w,c}}\} \\ &= \frac{1}{2}(\bar{p} + \underline{p}). \end{aligned}$$

Hence, $\eta(v_{A^{w,c}}) = y(\frac{1}{2}(\bar{p} + \underline{p}))$. □

The next corollary now follows from Theorem 5.3.13 and 5.3.14.

Corollary 5.3.15 *Let $(N^1, N^2, A^{w,c})$ be a square Böhm-Bawerk assignment situation, let $\bar{N}$ be such that $|\bar{N}| = |N^1|$ and take $\sigma^1 \in \Pi(\bar{N}, N^1)$, $\sigma^2 \in \Pi(\bar{N}, N^2)$. Then for the Böhm-Bawerk permutation situation $(\bar{N}, \bar{N}, \overline{A^{w,c}})$ it holds that $\eta(v_{\overline{A^{w,c}}}) = m^{\sigma^1\sigma^2}(\eta(v_{A^{w,c}})$.*

To obtain the results of Theorem 5.3.14 and Corollary 5.3.15, we suggest as another proof approach to use the results by Solymosi et al. (2005) on cyclic permutation situations and the pair-nucleolus and the proof that Núñez and Rafels (2005) provide for Theorem 5.3.13.

5.4 Permutation situations and games

In the last part of this chapter, our main focus is on permutation situations. We analyze the set of undominated matrices for permutation situations: given a permutation situation, the set of undominated matrices is formed by those matrices leading to the same permutation game for which we cannot raise any entry of the matrix without changing the core. Martínez-Albéniz et al. (2011) characterized the set of undominated matrices for assignment situations. Before we present our results on the set of undominated matrices for permutation situations, we will briefly repeat their results. Second, we discuss a subclass of the class of permutation situations. For these permutation situations, we characterize optimal matchings and provide explicit expressions for both the core and the nucleolus.

5.4.1 The core and exactness

First, we introduce two sets of matrices.

Definition 5.4.1 Let (N^1, N^2, A) be a matching situation. Then the set of matrices generating $C(v_A)$ is given by $G(A) = \{B \in \mathcal{M}^+_{N^1 \times N^2} \mid C(v_A) = C(v_B)\}$.

For $A, B \in \mathcal{M}^+_{N^1 \times N^2}$, we denote $A \geq B$ if $a_{ij} \geq b_{ij}$ for every $i \in N^1$, $j \in N^2$.

Definition 5.4.2 Let (N^1, N^2, A) be a matching situation. Then the set of *undominated matrices* generating $C(v_A)$ is given by $U(A) = \{B \in G(A) \mid \nexists C \in G(A)\backslash\{B\}$ such that $C \geq B\}$.

We illustrate the above definitions with the following example.

Example 5.4.3 Let (N, N, A) and $(N, N, B(\alpha, \beta))$ be permutation situations, where

$$A: \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \quad \text{and} \quad B(\alpha,\beta): \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix} \end{array},$$

for every $\alpha, \beta \in \mathbb{R}_+$. We have $C(v_A) = \{(1, 1)\}$. For a balanced 2-person game (N, v) it holds that

$$C(v) = \text{conv}\{(v(\{1\}), v(\{1, 2\}) - v(\{1\})), (v(\{1, 2\}) - v(\{2\}), v(\{2\}))\}.$$

So, for every game (N, v) such that $C(v) = C(v_A)$ it must hold that $v(\{1\}) = v(\{2\}) = 1$ and $v(\{1, 2\}) = 2$. Hence,

$$G(A) = \{B(\alpha, \beta) \mid \alpha + \beta \leq 2 \text{ and } \alpha, \beta \geq 0\}$$

and

$$U(A) = \{B(\alpha, \beta) \mid \alpha + \beta = 2 \text{ and } \alpha, \beta \geq 0\}.$$

◁

For every assignment situation (N^1, N^2, A), Martínez-Albéniz et al. (2011) characterize both $G(A)$ and $U(A)$. In fact, they show that $G(A)$ consists of a finite union of convex sets. In case that the optimal matching for the grand coalition is unique, $G(A)$ consists of one convex set. Also, they show that $U(A)$ is a singleton, and the unique element of $U(A)$ is the unique matrix in $G(A)$ that corresponds with a buyer-seller exact assignment game. Here, an assignment game $(N^1 \cup N^2, v_A)$ is called *buyer-seller exact* (Núñez and Rafels (2002)) if for every $i \in N^1, j \in N^2$, there exists a $(u, v) \in C(v_A)$ such that $u_i + v_j = v_A(\{i, j\})$.

Note that there does not need to exist a matrix $B \in G(A)$ such that $(N^1 \cup N^2, v_B)$ is exact, as the following example demonstrates.

Example 5.4.4 (Núñez and Rafels (2002)) Consider the assignment situation (N^1, N^2, A), where $N^1 = \{1, 2, 3\}$, $N^2 = \{4, 5, 6\}$ and A is given by

$$A : \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 5 & 8 & 3 \\ 7 & 9 & 6 \\ 2 & 3 & 0 \end{pmatrix} \end{array}.$$

The core of the assignment game (N, v_A) is given by $C(v_A) = \{((3, 5, 0), (2, 5, 1)), ((5, 6, 1), (1, 3, 0)), ((3, 6, 0), (2, 5, 0)), ((4, 6, 1), (1, 4, 0)), ((5, 6, 0), (2, 3, 0)), ((4, 5, 0), (2, 4, 1))\}$. It is readily checked that (N, v_A) is buyer-seller exact, hence, $U(A) = \{A\}$. As every exact assignment game is buyer-seller exact, this also means that if there exists a matrix $B \in G(A)$ such that $(N^1 \cup N^2, v_B)$ is exact, it must hold that $B = A$. However, $v_A(\{1\}) = 0$ but $\min_{(u,v) \in C(v_A)} u_1 = 3$. Hence, (N, v_A) is not exact. ◁

For every permutation situation (N, N, A) with $|N| \leq 3$ however, we can show that not only there exists a matrix $B \in U(A)$ such that (N, v_B) is exact, but this holds for every matrix $B \in U(A)$.

For every permutation situation (N, N, A) and every $i, j \in N$, define

$$\mathcal{S}_A^{ij} = \left\{ S \in 2^N \mid v_A(S) = \min_{y \in C(v_A)} y(S), \text{ and } \mu_{SS}^*(i) = j \text{ for some } \mu_{SS}^* \in M_A^*(S, S) \right\}.$$

So, if $S \in \mathcal{S}_A^{ij}$, then raising the entry a_{ij} would raise $v_A(S)$ and change the core of (N, v_A).

Theorem 5.4.5 Let (N, N, A) be a permutation situation, with $n \leq 3$. Then (N, v_B) is an exact game for all $B \in U(A)$.

Proof: The cases where $n = 1$ and $n = 2$ are trivial, as for $n < 3$ every balanced game is exact. Hence, let $n = 3$ and suppose that (N, v_B) is not exact for some $B \in U(A)$, i.e., there is $U \subsetneq N$ such that $v_B(U) < \min_{x \in C(v_B)} x(U)$. We distinguish two cases.

Case 1: $|U| = 1$.

Define $B^r \in \mathcal{M}^+_{N^1 \times N^2}$ by

$$\begin{array}{rclr} b^r_{ij} & = & b_{ij} & \text{for all } i \in N, j \in N\backslash\{i\}, \\ b^r_{ii} & = & \min_{x \in C(v_B)} x_i & \text{for all } i \in N. \end{array}$$

Observe that $B^r \geq B$, as $b_{ii} \leq v_B(\{i\})$ for every $i \in N$ with at least one strict inequality.
We show that $C(v_{B^r}) = C(v_B)$, which contradicts $B \in U(A)$. First we show $C(v_{B^r}) \subseteq C(v_B)$. Trivially,

$$v_{B^r}(S) \geq v_B(S) \text{ for all } S \in 2^N. \tag{5.2}$$

Let $\mu_{NN} \in \mathcal{M}^*_{B^r}(N, N)$. We take $S, T \in 2^N$ such that $S \cup T = N$, $\mu_{N,N}(i) = i$ for every $i \in S$ and $\mu_{N,N}(i) \neq i$ for every $i \in T$. Then

$$\begin{aligned} v_{B^r}(N) &= \sum_{i \in S} b^r_{i\mu_{NN}(i)} + \sum_{i \in T} b^r_{i\mu_{NN}(i)} \\ &\leq \sum_{i \in S} \min_{x \in C(v_B)} x_i + v_B(T) \\ &\leq \min_{x \in C(v_B)} \{x(S)\} + \min_{x \in C(v_B)} \{x(T)\} \\ &\leq \min_{x \in C(v_B)} x(N) = v_B(N) \leq v_{B^r}(N). \end{aligned}$$

Thus $v_{B^r}(N) = v_B(N)$, which, together with (5.2), implies that $C(v_{B^r}) \subseteq C(v_B)$. Next we show $C(v_B) \subseteq C(v_{B^r})$. Let $x \in C(v_B)$. By construction of B^r, $x_i \geq v_{B^r}(\{i\})$ for every $i \in N$. Take $i \in N$, $j \in N\backslash\{i\}$, take $S = \{i,j\}$ and $\mu_{SS} \in \mathcal{M}^*_{B^r}(S,S)$. If $\mu_{SS}(i) = i$ and $\mu_{SS}(j) = j$ then by definition of b^r_{ii} and b^r_{jj},

$$v_{B^r}(S) = b^r_{ii} + b^r_{jj} \leq x_i + x_j.$$

If $\mu_{SS}(i) = j$ and $\mu_{SS}(j) = i$ then by construction of B^r,

$$v_{B^r}(S) = b_{ij} + b_{ji} = v_B(S) \leq x_i + x_j,$$

as $x \in C(v_B)$. Therefore, $C(v_{B^r}) = C(v_B) = C(v_A)$. In conclusion, $B^r \in U(A)$, $B^r \geq B$ and $B^r \neq B$, which is a contradiction with $B \in U(A)$.

Case 2: $|U| = 2$.

Take $i,j \in N$ such that $U = \{i,j\}$ and let $k \in N\backslash\{i,j\}$. By assumption,

$$v_B(U) = \max\{b_{ii} + b_{jj}, b_{ij} + b_{ji}\} < \min_{x \in C(v_B)} x_i + x_j. \tag{5.3}$$

Since $B \in U(A)$, we cannot increase b_{ij} and b_{ji} without changing the core of the corresponding permutation game. Hence, there exists an $S^{ij} \in \mathcal{S}^{ij}_B$. Since $v_B(U) < \min_{x \in C(v_B)}(x_i + x_j)$ and $n = 3$, we necessarily have $S^{(i,j)} = \{i,j,k\}$. Note that $v_B(N) = b_{ij} + b_{ji} + b_{kk}$ would imply that $x_k = b_{kk}$ for all $x \in C(v_B)$. This implies $v_B(U) = x_i + x_j = \min_{z \in C(v_B)} z(U)$, which contradicts (5.3). Therefore

$$v_B(N) = b_{ij} + b_{jk} + b_{ki} = x(N), \tag{5.4}$$

for all $x \in C(v_B)$.
Analogously, with $S^{ji} \in \mathcal{S}^{ij}$ we obtain

$$v_B(N) = b_{ji} + b_{ik} + b_{kj} = x(N), \tag{5.5}$$

for all $x \in C(v_B)$. From (5.4) and (5.5), for all $x \in C(v_B)$,

$$\begin{aligned}
2v_B(N) &= (b_{ij} + b_{jk} + b_{ki}) + (b_{ji} + b_{ik} + b_{kj}) \\
&= (b_{ij} + b_{ji}) + (b_{ik} + b_{ki}) + (b_{jk} + b_{kj}) \\
&\leq v_B(\{i,j\}) + v_B(\{i,k\}) + v_B(\{j,k\}) \\
&< 2x(N) \\
&= 2v_B(N),
\end{aligned}$$

a contradiction. Note that the strict inequality holds by (5.3) and the last inequality holds since $x \in C(v_B)$. □

The proof of the above result could be simplified a little bit by further exploiting the richer structure of the three-player set case. However, we have chosen this more general proof for it could guide the reader to understand (and maybe solve) the general case. Because of the unsuccessful search of counterexamples we conjecture that the result of Theorem 5.4.5 may hold in the general case, i.e, for an arbitrary player set. Note that for games with at most three players, every exactness is equivalent with convexity. Hence, every undominated matrix results in a convex game. We conclude this part with the following example, which shows that $U(A)$ can consist of more than one convex set.

Example 5.4.6 Consider the permutation situations (N, N, A) and (N, N, B) where $N = \{1, 2, 3\}$ and A and B are given by

$$A: \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 3 \\ 3 & 0 & 0 \end{pmatrix} \end{array} \text{ and } B: \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 0 & 3 \\ 3 & 2 & 0 \\ 0 & 3 & 0 \end{pmatrix} \end{array}.$$

These matrices both lead to the following permutation game

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v_A(S)$	1	2	0	3	3	3	9

with $C(v) = \text{conv}\{(1, 2, 6), (1, 6, 2), (3, 6, 0), (6, 2, 1), (6, 3, 0)\}$. Note that (N, v_A) is exact, so we cannot raise the value of any coalition without changing the core.

Any matrix $C \in G(A)$ must satisfy the following inequalities:

$$\begin{array}{ll} c_{11} \leq 1 & c_{12} + c_{21} \leq 3 \\ c_{22} \leq 2 & c_{13} + c_{31} \leq 3 \\ c_{33} \leq 0 & c_{23} + c_{32} \leq 3 \end{array}$$

As $v_A(N) = 9$, these conditions imply that no entry of C is larger than 3, and that no entry on the diagonal can be used in an optimal matching for N. Hence, either $c_{12} = c_{23} = c_{31} = 3$ (and therefore $c_{21} = c_{32} = c_{13} = 0$) or $c_{13} = c_{32} = c_{21} = 3$ (and therefore $c_{31} = c_{23} = c_{12} = 0$). Also, to ensure that $(1, 2, 6) \in \text{ext}(C(v_C))$ and $(1 + a, 2 - a, 6) \notin C(v_C)$ for every $a \in [-1, 0) \cup (0, 2]$, it must hold that $c_{11} = 1$ and $c_{22} = 2$. Together, this shows that $G(A) = G(B) = \{A, B\}$. Also, for both A and B, every entry is used in an optimal matching for some coalition $S \in N$: e.g., the entry $a_{(1,3)}$ is used in the optimal matching for coalition $\{1, 3\}$. Hence, the matrices A and B are undominated: $U(A) = U(B) = \{A, B\}$. Note that $B = A^\top$. In fact it is easily seen that if a matrix is undominated, then also its transpose is undominated. $\triangleleft$

5.4.2 'Homogeneous alternatives' permutation

Now, we analyze a subclass of the class of permutation situations. In this class of permutation situations, every player considers the objects of the other players to be homogeneous. So, a player either prefers the status quo or prefers any other object to his own. This can be represented by two vectors $\alpha, \beta \in \mathbb{R}^N_+$ such that α_i represents the valuation of player $i \in N$ of his own object, and β_i represents the valuation of player $i \in N$ of the other objects.

Definition 5.4.7 Let $\alpha, \beta \in \mathbb{R}^N_+$. Then the *'homogeneous alternatives'* (HA) permutation situation $(N, N, A^{\alpha\beta})$ is given by

$$a_{(i,j)} = \begin{cases} \alpha_i & \text{if } i = j, \\ \beta_i & \text{if } i \neq j, \end{cases} \tag{5.6}$$

for every $i, j \in N$.

To be able to provide expressions for the core of the permutation game, we first characterize the optimal matchings. To this end, we first introduce some notation. Let $(N, N, A^{\alpha\beta})$ be a HA-permutation situation. We define two sets: $I = \{i \in N \mid \beta_i < \alpha_i\}$ and $C = \{i \in N \mid \beta_i \geq \alpha_i\}$. The set I represents the 'individuals': those players that strictly prefer to keep their own object. The set C on the other hand represents the 'cooperators', being the players that would profit from switching objects with another player. For these sets, we take $\bar{\beta} = \min\{\beta_k \mid k \in C\}$ and

$\bar{\alpha} = \min\{\alpha_i \mid i \in I\}$. We define two special permutations: $\mu^0_{NN} \in M(N,N)$ is such that for every $i \in N$, $\mu^0_{NN}(i) = i$. Furthermore, for $i \in N$, $j \in N\backslash\{i\}$, we define $\mu^{(i,j)}_{NN} \in M(N,N)$ such that $\mu^{(i,j)}_{NN}(i) = j$, $\mu^{(i,j)}_{NN}(j) = i$ and $\mu^{(i,j)}_{NN}(k) = k$ for every $k \in N\backslash\{i,j\}$. So, μ^0_{NN} is the permutation that assigns everyone his own object, and $\mu^{(i,j)}_{NN}$ is the permutation where only player i and j switch objects. Also, let $I^* \subseteq I$ be given by $I^* = \{i^* \in I \mid \alpha_{i^*} - \beta_{i^*} \leq \alpha_i - \beta_i \text{ for every } i \in I\}$. So, I^* contains those players of I that are most willing to switch objects. Lastly, for every $S \in 2^N$, let $d_S \in \mathbb{R}$ be given by

$$d_S = \begin{cases} \min\limits_{i\in S}\{\alpha_i - \beta_i\} & \text{if } S \neq \emptyset \\ 0 & \text{if } S = \emptyset \end{cases}$$

Theorem 5.4.8 Let $(N, N, A^{\alpha\beta})$ be a HA-permutation situation such that $|C| \neq 1$. Then $\mu_{NN} \in M^*_{A^{\alpha\beta}}(N,N)$ if and only if $\mu_{NN}(i) = i$ for every $i \in I$ and $\mu_{NN}(k) \in C\backslash\{k\}$ for all $k \in C$ such that $\beta_k - \alpha_k > 0$.

Proof: Let $\mu_{NN} \in M(N,N)$ be such that $\mu_{NN}(i) = i$ for every $i \in I$ and $\mu_{NN}(k) \in C\backslash\{k\}$ for all $k \in C$ such that $\beta_k - \alpha_k > 0$. As $|C| \neq 1$, such a μ_{NN} exists. We have for every $i \in N$, $a_{(i,\mu_{NN}(i))} \geq a_{(i,j)}$ for every $j \in N$. Hence, $\sum_{i\in N} a_{(i,\mu_{NN}(i))} \geq \sum_{i\in N} a_{(i,\mu'_{NN}(i))}$ for every $\mu'_{NN} \in M(N,N)$. Hence, $\mu_{NN} \in M^*_{A^{\alpha\beta}}(N,N)$.
For every $\mu'_{NN} \in M(N,N)$ that does not satisfy the conditions in the theorem, it holds that $a_{(i,\mu'_{NN}(i))} < a_{(i,\mu_{NN}(i))}$ for some $i \in N$, and $a_{(j,\mu'_{NN}(j))} \leq a_{(j,\mu_{NN}(j))}$ for all $j \in N\backslash\{i\}$. Then, $\sum_{i\in N} a_{(i,\mu_{NN}(i)} < \sum_{i\in N} a_{(i,\mu'_{NN}(i))}$ and μ'_{NN} is not optimal. $\square$

Theorem 5.4.9 Let $(N, N, A^{\alpha\beta})$ be an HA-permutation situation such that $|C| = 1$ and $n \geq 2$. Then

$$M^*_{A^{\alpha\beta}}(N,N) = \begin{cases} \{\mu^0_{NN}\} & \text{if } d_I > \beta_k - \alpha_k, \\ \{\mu^0_{NN}\} \cup \{\mu^{(i^*,k)}_{NN}\}_{i^*\in I^*} & \text{if } d_I = \beta_k - \alpha_k, \\ \{\mu^{(i^*,k)}_{NN}\}_{i^*\in I^*} & \text{if } d_I < \beta_k - \alpha_k, \end{cases}$$

where $k \in C$.

Proof: Let $\mu_{NN} \in M(N,N)$. We have $\sum_{i\in N} a_{i\mu_{NN}(i)} \leq \sum_{i\in N\backslash\{k\}} \alpha_i + \max\{\alpha_k, \beta_k - d_I\}$, with equality if and only if $a_{(i^*,\mu_{NN}(i^*))} + a_{(k,\mu_{NN}(k))} = \max\{\alpha_k + \alpha_{i^*}, \beta_k + \beta_{i^*}\}$, where $i^* \in I^*$. This implies the stated expression for $M^*_{A^{\alpha\beta}}(N,N)$. $\square$

Now, we will provide explicit expressions for the core of the HA-permutation game. Restricting ourselves here to HA-matrices adds a lot of structure. In several cases, the core is a singleton.

Theorem 5.4.10 Let $(N, N, A^{\alpha\beta})$ be an HA-permutation situation such that either $|C| \geq 3$ or $|C| = 0$. Then $C(v_A) = \{x\}$ where

$$x_i = \begin{cases} \alpha_i & \text{if } i \in I, \\ \beta_i & \text{if } i \in C, \end{cases}$$

for every $i \in N$.

Proof: If $|C| = 0$ then $M^*_{A^{\alpha\beta}}(N, N) = \{\mu^0_{NN}\}$ and therefore $v_A(\{i\}) + v_A(N \backslash \{i\}) = v_A(N)$ for every $i \in N$. This implies that $C(v_A) = \{x\}$ where $x_i = v_A(\{i\}) = \alpha_i$ for every $i \in N$.
If $|C| \geq 3$, then for $i \in I$ it holds that $v_A(\{i\}) + v_A(N \backslash \{i\}) = v_A(N)$, so $x_i = \alpha_i$ for every $i \in I$. Take $k \in C$ and divide $C \backslash \{k\}$ into two arbitrary non-empty sets C^1 and C^2. As $|C| \geq 3$, this is possible. Since $v_A(C) = x(C) = \sum_{k \in C} \beta_k$ for every $x \in C(v_A)$ we obtain via $x(C) + x_k \geq v_A(C^1 \cup \{k\}) + v_A(C^2 \cup \{k\}) = \sum_{l \in C^1} \beta_l + \beta_k + \sum_{l \in C^2} \beta_l + \beta_k = \sum_{l \in C} \beta_l + \beta_k$ that $x_k = \beta_k$ for every $k \in C$. Hence, $C(v_A) = \{x\}$ where $x_i = \alpha_i$ for every $i \in I$ and $x_k = \beta_k$ for every $k \in C$. □

In the case where $|C| = 2$, the core of the permutation game is a segment. Still, for every player in I the allocation is equal across all elements of the core, but the players in C can share the profit from exchanging their objects in more than one way.

Theorem 5.4.11 Let $(N, N, A^{\alpha\beta})$ be an HA-permutation situation such that $|C| = 2$. Then $C(v_A) = \text{conv}\{c^k \in \mathbb{R}^N \mid k \in C\}$, where, for all $k \in C$ and $l \in N$,

$$c^k_l = \begin{cases} \alpha_l & \text{if } l \in I, \\ \max\{\alpha_k, \beta_k - d_I\} & \text{if } l = k, \\ \beta_k + \beta_l - \max\{\alpha_k, \beta_k - d_I\} & \text{if } l \in C \backslash \{k\}. \end{cases}$$

Proof: Assume $|I| = 0$. It is readily checked that the standard expression for the core of a 2-person game reduces to the provided expression for the core. Now assume $|I| > 0$.

First, we show that $C(v_A) \subseteq \text{conv}\{c^k \in \mathbb{R}^N \mid k \in C\}$. Let $\mu^*_{N,N} \in M^*_{A^{\alpha\beta}}(N,N)$ and let $x \in C(v_A)$. Since $\mu^*_{N,N}(i) = i$ for all $i \in I$, we obtain that $v_A(\{i\}) + v_A(N\backslash\{i\}) = v(N)$, and therefore $x_i = v_A(\{i\})$. Let $C = \{l, k\}$. We have $\mu^*_{N,N}(l) = k$ and $\mu^*_{N,N}(k) = l$. So, $x_l + x_k = \beta_l + \beta_k$. Via $v_A(i,k) = \max\{\alpha_i + \alpha_k, \beta_i + \beta_k\}$ we obtain $x_k \geq \max\{\alpha_k, \beta_i + \beta_k - \alpha_i\}$ for every $k \in C$ and $i \in I$. Hence, $x_k \geq \max\{\alpha_k, \beta_k - d_I\}$. So, $C(v_A) \subseteq \text{conv}\{c^k \in \mathbb{R}^N \mid k \in C\}$.
On the other hand, it is easily verified that $\text{conv}\{c^k \in \mathbb{R}^N \mid k \in C\} \subseteq C(v_A)$. Hence, $C(v_A) = \text{conv}\{c^k \in \mathbb{R}^N \mid k \in C\}$. □

If $|C| = 1$, there are two distinct situations. In the first case, it is not optimal for a player in I to exchange objects with the player in C, so the core is a singleton. The second case is in fact similar to Theorem 5.4.11, where the player in I^* exchanges his object with the player in C instead of - the now non-existent - second player in C.

Theorem 5.4.12 Let $(N, N, A^{\alpha\beta})$ be an HA-permutation situation such that $|C| = 1$ and $|I| > 0$. If $|I^*| > 1$, or $d_I \geq \beta_k - \alpha_k$ for $k \in C$, then $C(v_A) = \{y\}$ where

$$y_i = \begin{cases} \alpha_i & \text{if } i \in I, \text{ or } i \in C \text{ and } d_I \geq \beta_i - \alpha_i \\ \beta_i - \min\{\alpha_j - \beta_j \mid j \in I\} & \text{if } i \in C \text{ and } |I^*| > 1. \end{cases}$$

Proof: Assume $|C| = 1$ and $d_I \geq \beta_k - \alpha_k$, let $x \in C(v_A)$. By Theorem 5.4.9, we have $\mu^0_{NN} \in M^*_{A^{\alpha\beta}}(N,N)$. Hence, for every $i \in N$ we have $v_A(\{i\}) + v_A(N\backslash\{i\}) = v_A(N)$ and therefore $x_i = v_A(\{i\}) = y_i$ for every $i \in N$. Hence, $C(v_A) \subseteq \{y\}$. As the core of a permutation game is non-empty, it follows that $C(v_A) = \{y\}$.
Now take $d_I < \beta_k - \alpha_k$ and $I^* > 1$, let $x \in C(v_A)$. The implication of $|I^*| > 1$ is that $(N\backslash\{i\}) \cap I^* \neq \emptyset$ for every $i \in I$. This means that $v_A(N\backslash\{i\}) = \sum_{j \in N\backslash\{i\}} \alpha_j - d_I + \beta_k$ for every $i \in I$. Therefore, for every $i \in I$ it holds $v_A(\{i\}) + v_A(N\backslash\{i\}) = v_A(N)$ and $x_i = v_A(\{i\}) = y_i$ for every $i \in N$. This means $x_k = \beta_k - d_I = y_k$. Hence, $C(v_A) \subseteq \{y\}$. As the core of a permutation game is non-empty, it follows that $C(v_A) = \{y\}$. □

Theorem 5.4.13 Let $(N, N, A^{\alpha\beta})$ be an HA-permutation situation such that $|C| = 1$, $|I| > 0$, $|I^*| = 1$ and $d_I < \beta_k - \alpha_k$ for $k \in C$. Then $x \in C(v_A)$ if and only if

$x_i = \alpha_i$ for all $i \in I \backslash I^*$, $x_{i^*} \in [\alpha_{i^*}, \beta_k + \beta_{i^*} - \max\{\alpha_k, \beta_k - \min\{\alpha_i - \beta_i \mid i \in I \backslash I^*\}\}]$ and $x_k = \beta_i + \beta_k - x_{i^*}$ for $i^* \in I^*$ and $k \in C$.

Proof: For the 'if'-part, it is readily checked that $\sum_{i \in S} x_i \geq v_A(S)$ for every $S \subseteq N$.
For the 'only if'-part, assume $x \in C(v_A)$. Take $i^* \in I^*$ and $k \in C$. We have that $\mu_{NN}(i) = i$, and therefore $x_i = \alpha_i$ for every $i \in I \backslash I^*$. As $x_k \geq v_A(\{i, k\}) - x_i = \max\{\alpha_i + \alpha_k, \beta_i + \beta_k\} - \alpha_i = \max\{\alpha_k, \beta_i + \beta_k - \alpha_i\}$ for every $i \in I \backslash I^*$, we obtain via $x_{i^*} \geq \alpha_{i^*}$ and $x_{i^*} + x_k = v_A(\{i^*, k\}) = \beta_k + \beta_{i^*}$ that $x_{i^*} \in [\alpha_i, \beta_k + \beta_i - \max\{\alpha_l, \max_{j \in I \backslash D_I} \beta_j - \alpha_j + \beta_k\}]$. $\square$

The following theorem shows that for every HA-permutation game, the nucleolus coincides with the barycenter of the core.

Theorem 5.4.14 Let (N, v_A) be a HA-permutation game. Then $\eta(v_A)$ is the barycenter of the core.

Proof: For those cases where $C(v_A)$ is a singleton, it is trivial that $\eta(v_A)$ equals the barycenter of the core. Also, if $|N| = 2$ the barycenter of the core and $\eta(v_A)$ coincide. So, assume $|N| > 2$. Now two cases where the core is not a singleton remain:

- $|C| = 2$. As $|N| > 2$, $|I| > 0$. Let $x \in C(v_A)$. First, we determine which coalitions are relevant for determining the nucleolus. For every coalition $S \in 2^N$ such that either $C \cap S = C$ or $C \cap S = \emptyset$, we have $E(S, x) = 0$. Now take $k, l \in C$, $k \neq l$ and $S \subseteq I$. We have $E(\{k\}, x) = \alpha_k - x_k$, and $E(S \cup \{k\}) = \max\{\alpha_k, \beta_k - d_S\} - x_k$. Hence,

$$\begin{aligned} E(\{k\}, x) &\leq E(\{S \cup \{k\}, x\}) \\ &\leq E(\{i^*, k\}, x) \\ &= \max\{\alpha_k, \beta_k - d_I\} - x_k, \end{aligned}$$

  where $i^* \in I^*$. Similarly, we obtain $E(\{l\}, x) \leq E(S \cup \{l\}, x) \leq E(\{i^*, l\}, x) = \max\{\alpha_l, \beta_l - d_I\} - x_l$. Hence, the highest excess that is not constant across all core elements is attained for either coalition $\{i^*, k\}$ or coalition $\{i^*, l\}$. Since

$$\begin{aligned} E(\{i^*,k\},x)+E(\{i^*,l\},x) &= \max\{\alpha_k,\beta_k-d_I\} \\ &\quad +\max\{\alpha_l,\beta_l-d_I\} \\ &\quad -(\beta_k+\beta_l), \end{aligned}$$

is independent of x, the excess vector of x is lexicographically minimized if $E(\{i^*,k\},x)=E(\{i^*,l\},x)$. Hence,

$$\begin{aligned} E(\{i^*,k\},\eta(v_A)) &= E(\{i^*,l\},\eta(v_A)) \\ &= \frac{1}{2}\left(\max\{\alpha_k,\beta_k-d_I\}\right. \\ &\quad \left.+\max\{\alpha_l,\beta_l-d_I\}-(\beta_k+\beta_l)\right), \end{aligned}$$

which gives:

$$\begin{aligned} \eta_k(v_A) &= v_A(\{i^*,k\})-\eta_{i^*}(v_A)-E(\{i^*,k\},\eta(v_A)) \\ &= \max\{\alpha_k+\alpha_{i^*},\beta_k+\beta_{i^*}\}-\alpha_{i^*} \\ &\quad -\frac{1}{2}\left(\max\{\alpha_k,\beta_k-d_I\}\right. \\ &\quad \left.+\max\{\alpha_l,\beta_l-d_I\}-(\beta_k+\beta_l)\right) \\ &= \frac{1}{2}\left(\max\{\alpha_k,\beta_k-d_I\}\right. \\ &\quad \left.+(\beta_k+\beta_l)-\max\{\alpha_l,\beta_l-d_I\}\right) \end{aligned}$$

and $\eta_l(v)=(\beta_k+\beta_l)-\eta_k(v)$. By Theorem 5.4.11 we have that this is the barycenter of the core.

- $|C|=1$, $|I|>0$ and $|I^*|=1$. As $|N|>2$, $|I|>1$. Let $x\in C(v_A)$ and let $i^*\in I^*$ and $k\in C$. First, we determine which coalitions are relevant for determining the nucleolus. For every coalition $S\in 2^N$ such that either $\{i^*,k\}\cap S=S$ or $\{i^*,k\}\cap S=\emptyset$, we have $E(S,x)=0$. Now take $S\subseteq N\backslash\{i^*,k\}$, $S\neq\emptyset$. We have $E(\{i^*\},x)=E(S\cup\{i^*\},x)=\alpha_{i^*}-x_{i^*}$, $E(\{k\},x)=\alpha_k-x_k$, and

$E(S \cup \{k\}) = \max\{\alpha_k, \beta_k - d_S\} - x_k$. Hence,

$$\begin{aligned} E(\{k\},x) &\leq E(\{S \cup \{k\},x\}) \\ &\leq E(\{\bar{i},k\},x) \\ &= \max\{\alpha_k, \beta_k - d_{I \setminus \{i^*\}}\} - x_k, \end{aligned}$$

where $\bar{i} \in \{i \in I \setminus \{i^*\} \mid \alpha_i - \beta_i \leq \alpha_j - \beta_j$ for every $j \in I \setminus \{i^*\}\}$. Hence, the highest excess that is not constant across all core elements is attained for either coalition $\{i^*\}$ or coalition $\{\bar{i}, k\}$.

Since $x_{i^*} + x_k = \beta_{i^*} + \beta_k$,

$$E(\{\bar{i},k\},x) + E(\{i^*\},x) = \max\{\alpha_k, \beta_k - (\alpha_{\bar{i}} - \beta_{\bar{i}})\} + \alpha_{i^*} - (\beta_k + \beta_{i^*})$$

is independent of x. Therefore the excess vector of x is lexicographically minimized if $E(\{i^*\},x) = E(\{\bar{i},k\},x)$, so $E(\{\bar{i}\},\eta(v)) = E(\{i^*,k\},\eta(v)) = \frac{1}{2}(\max\{\alpha_k, \beta_k - (\alpha_{\bar{i}} - \beta_{\bar{i}})\} + \alpha_{i^*} - (\beta_k + \beta_{i^*}))$. This gives:

$$\begin{aligned} \eta_{i^*}(v_A) &= v_A(\{i^*\}) - E(\{i^*\},\eta(v_A)) \\ &= \alpha_{i^*} - \frac{1}{2}\left(\max\{\alpha_k, \beta_k - (\alpha_{\bar{i}} - \beta_{\bar{i}})\} + \alpha_{i^*} - (\beta_k + \beta_{i^*})\right) \\ &= \frac{1}{2}\left(\alpha_{i^*} + (\beta_k + \beta_{i^*}) - \max\{\alpha_k, \beta_k - \alpha_{\bar{i}} + \beta_{\bar{i}}\}\right), \end{aligned}$$

and $\eta_k(v) = (\beta_k + \beta_l - \eta_{i^*}(v))$. By Theorem 5.4.13 we have that this is the barycenter of the core. $\square$

Chapter 6

Sequencing situations with Just-in-Time arrival, and related games

6.1 Introduction

Sequencing theory deals with a variety of problems sharing several characteristics: a number of jobs have to be processed on one or more machines, in such a way that a cost criterion is minimized. From one sequencing problem to another the way these characteristics are defined can differ and additional constraints can be added: the machines can be parallel or serial, there can be conditions on the order in which the jobs should be processed and different cost criteria can be used. Applications of the theory of sequencing situations are numerous and diverse: from manufacturing and maintenance to scheduling patients in an operating room.

The starting point of the game theoretic analysis of sequencing situations is the paper by Curiel, Pederzoli, and Tijs (1989). In their one-machine model, only one job can be processed at a time. The processing time is deterministic for every job, and every job has a certain constant cost per time unit it spends in the system. A job is in the system from the moment the machine starts processing the first job until the job itself is processed by the machine. An order that minimizes total cost, processes the jobs in a decreasing order with respect to their urgency (cost per time unit divided by the length of the job, cf. Smith (1956)). A procedure is introduced that, given an initial order, uses neighbor switches to obtain the optimal order and constructs a stable cost allocation in the process. Since Curiel et al. (1989) several related classes of sequencing problems are discussed, including ready times, due dates, multiple machines and numerous cost criteria (see e.g., Curiel, Hamers, and Klijn (2002), Borm, Fiestras-Janeiro, Hamers, Sanchez, and Voorneveld (2002), Calleja, Borm, Hamers, Klijn, and Slikker (2002) and Slikker (2005) for game-theoretic discussions).

From an operations research point of view, classes of sequencing problems where between jobs one needs time to set-up the machine are discussed extensively in the literature. Different types of set-up time are considered to match the application under consideration, such as sequencing aircraft landings (Psaraftis (1980)) and steel pipe manufacturing (Ahn and Hyun (1990)). In Gupta (1988) the mean flow time is minimized in sequencing situations with switching times between jobs depending on the class of both jobs. The change-over model by Van der Veen et al. (1998) on the other hand minimizes the makespan in a setting which uses set-up times as well as after-processing times to define the switching time. Çiftçi (2009) discusses a sequencing model where a predecessor-independent switching time occurs if two subsequent jobs belong to different classes of jobs. Here, the focus is on the game-theoretic aspect of cost-sharing, based on the presence of an initial queue.

Similar to Gupta (1988) and Van der Veen, Woeginger, and Zhang (1998), we incorporate a set-up time in the model that is to be executed by the job to be processed next. We consider a basic setting of this set-up time - one can think of cleaning up a machine, adjusting a machine to the new jobs or something simple as erasing the blackboard before one can start the lecture - such that it only depends on the state in which the system is left behind by the predecessor.

A key feature of the sequencing situations discussed in this chapter, which is based on Lohmann et al. (2010), is the Just-in-Time (JiT) arrival of the jobs: a job just arrives at the factory as soon as its predecessor is finished. Hence, we leave the setting of the sequencing literature described above, where every job is waiting in a queue from the moment the first job starts. The owner of the job has to execute the set-up himself. So, given the job that is processed before, the time it takes before a next job can start processing is fixed. The time a job spends in the factory is formed by this predecessor dependent set-up time and the length of the job itself. This way, the cost incurred by a job depends on the set-up time, its own processing time and his individual cost per time unit. As the costs incurred during the processing of the job itself is constant across all possible orders of jobs, we do not incorporate these costs into our analysis. For sequencing without set-ups, JiT arrival of the jobs would imply that the time a job spends in the system equals the processing time of the job itself. Of course, this problem is trivial as in this case every order of jobs results in the same costs.

Verdaasdonk (2007) initiated the study on this subject. The model discussed in this paper can also be modeled in terms of the traveling salesman problem (see e.g. Lawler et al. (1985)). Our specific costs structure makes that the matrix underlying the traveling salesman problem corresponding with a sequencing situation with JiT arrival in general is not contained in any well-studied matrix class for the traveling salesman problem such as the class of Monge matrices or Van der Veen matrices (see, e.g., Burkard et al. (1998)). This means that, to the best of our knowledge, for this specific subclass of traveling salesman problems no efficient algorithm is known in the literature. So, to obtain a manageable optimization problem we focus in the current chapter on those sequencing situations with JiT arrival and predecessor dependent set-up times, where there are two different values for the set-up time and two different values for the costs per time unit.

The topic will be treated from two perspectives. The first part concerns the operations research perspective. For each sequencing situation with JiT arrival (or, for short, JiT sequencing situation) we provide sufficient conditions to check if an order minimizes joint costs, and provide an algorithm to obtain such an optimal order. The second part, concerning the game theoretic perspective, involves the allocation of the minimal joint costs. For this, we define an objective and consistent way to determine cost savings for each coalition of jobs. In particular, we assume that the players are part of a larger system with players in the system before and after the grand coalition, so that for the grand coalition consistent with the worst-case approach for subcoalitions, the system is left behind with high set-up time due.

A first focus point in the game theoretic approach is the core. We show that every JiT sequencing game has a nonempty core. Furthermore, we provide explicit expressions for the core of a large class of JiT sequencing games. In particular, we formulate conditions such that the core of the JiT sequencing game is a singleton. Also, similarities between a special class of assignment games called Böhm-Bawerk horse market games (see e.g., Böhm-Bawerk (1891), Núñez and Rafels (2005) and Chapter 5 in this dissertation), and certain JiT sequencing games are pointed out. It is well known that the core of a Böhm-Bawerk horse market game is a line segment (Shapley and Shubik (1972)), where one extreme point is 'buyer'-optimal and the other extreme point is 'seller'-optimal. Under mild conditions, the core of a JiT sequencing game is a line segment, where we can identify two subsets of the player set, acting as the 'buyers' and the 'sellers'.

A second focus point is the nucleolus. As the nucleolus is a core-selector, for those JiT sequencing games where the core is a singleton the nucleolus coincides with the core. If the core is a line segment, we provide an explicit expression of the nucleolus in terms of the underlying JiT sequencing situation. As the general expression for the nucleolus heavily depends on the exact parameters, we provide a more basic and less volatile allocation rule called the *large instance based allocation rule* that coincides with the nucleolus for large classes of JiT sequencing games and is also contained in the core for every JiT sequencing game.

This chapter is organized as follows: in the subsequent section, we formally introduce the JiT sequencing model. Also, we provide optimality conditions regarding the processing order, and give an algorithm to find an optimal order. Section 6.3 contains the game theoretic analysis and focusses on characterizing the core and the nucleolus for JiT sequencing games.

6.2 JiT sequencing situations

A *JiT sequencing situation* is defined by a tuple $\Psi = (N, \alpha, s, s_0)$. Here, N denotes the nonempty finite player set. It is assumed that every player owns exactly one job. As there is a one-to-one correspondence between players and jobs, we will use the words player and job interchangeably throughout this chapter. The vector $\alpha \in \mathbb{R}^N_+$ is such that for player $i \in N$, the costs of spending t time units in the system is given by $\alpha_i t$. The *set-up times* are denoted by the vector $s \in \mathbb{R}^N_+$, where for $i \in N$, s_i is the set-up time needed after the job of player i is processed and before the machine can process another job. The time needed before the machine can process the first job is denoted by s_0. The order $\sigma \in \Pi(N)$ describes the *order of processing* of the jobs. For notational convenience, we set $\sigma(0) = 0$ and therefore $s_{\sigma(0)} = s_0$ for all $\sigma \in \Pi(N)$.

In JiT sequencing situations, it is assumed that a player enters the system at the moment the player starts to prepare the machine for his job and leaves the system as soon as his job is finished. This situation is shown in Figure 6.2.1. This differs from standard sequencing problems as depicted in Figure 6.2.2, where a player enters the system already as the first job in the order starts processing and leaves after his own job is finished. Also, we include set-up times. The time a job spends in the system consists of a set-up time depending on the job that is processed before him

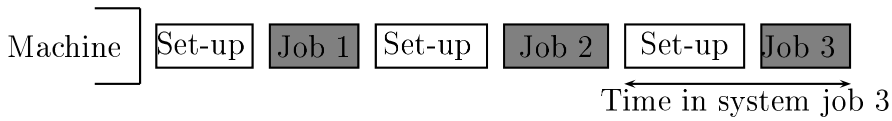

Figure 6.2.1: Time in system for JiT sequencing

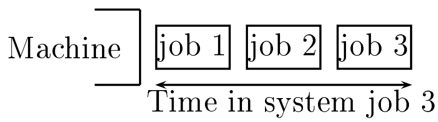

Figure 6.2.2: Time in system for standard sequencing

and his own processing time. The costs arising from this last part is constant over all orders. Hence, we just focus on the costs arising from set-up. So, for an order $\sigma \in \Pi(N)$ the corresponding costs $\gamma_i(\sigma)$ for player $i \in N$ are given by

$$\gamma_i(\sigma) = \alpha_i s_{\sigma(\sigma^{-1}(i)-1)}.$$

For a coalition $S \in 2^N$, we set $\gamma_S(\sigma) = \sum_{i \in S} \gamma_i(\sigma)$. We call an order $\sigma^* \in \Pi(N)$ optimal for N if $\gamma_N(\sigma^*) = \min\{\gamma_N(\sigma) \mid \sigma \in \Pi(N)\}$.

In this chapter we restrict ourselves to the analysis of JiT sequencing situations with two different values for the set-up times and two different values for the cost per time unit. We denote by $JiT^{2,2}$ the class of all JiT sequencing situations satisfying this restriction. So, for every $(N, \alpha, s, s_0) \in JiT^{2,2}$, there exist $\alpha^H, \alpha^L \in \mathbb{R}_+$, $\alpha^H > \alpha^L$ such that for all $i \in N$ it holds that either $\alpha_i = \alpha^H$ or $\alpha_i = \alpha^L$. With respect to the set-up times, we assume there exist $s^h, s^l \in \mathbb{R}_+$, $s^h > s^l$, such that for all $i \in N \cup \{0\}$ it holds that either $s_i = s^h$ or $s_i = s^l$. We partition the set of players according to their characteristics as provided in Table 6.2.1, into sets N_h^H, N_l^H, N_h^L and N_l^L. Note that the superscript refers to the cost per time unit, and the subscript refers to the set-up time. Also, throughout the chapter uppercase H and L refer to cost per time unit and lowercase h and l to set-up time. We denote $N^H = N_h^H \cup N_l^H$, and define N^L, N_h and N_l in a similar way. For a subset $S \in 2^N$ we use a similar notation: $S_h^H = S \cap N_h^H$, $S^H = S \cap N^H$, etc.

Example 6.2.1 Consider the JiT sequencing problem $\Psi = (N, \alpha, s, s_0)$, where $N = \{1, 2, 3\}$, $\alpha = (\alpha_1, \alpha_2, \alpha_3) = (4, 1, 1)$, $s = (s_1, s_2, s_3) = (3, 3, 1)$ and $s_0 = 3$.

	costs	*set-up time*
N_h^H	α^H	s^h
N_l^H	α^H	s^l
N_h^L	α^L	s^h
N_l^L	α^L	s^l

Table 6.2.1: Partition of the player set

It is readily checked that $N_h^H = \{1\}$, $N_l^H = \emptyset$, $N_h^L = \{2\}$ and $N_l^L = \{3\}$. The order $\sigma \in \Pi(N)$ such that $\sigma(1) = 3$, $\sigma(2) = 2$ and $\sigma(3) = 1$, as shown in Figure 6.2.3, gives $\gamma_1(\sigma) = s_2\alpha_1 = 12$, $\gamma_2(\sigma) = s_3\alpha_2 = 1$, and $\gamma_3(\sigma) = s_0\alpha_3 = 3$, so $\gamma_N(\sigma) = 16$. However, the order $\sigma' \in \Pi(N)$ such that $\sigma'(1) = 2$, $\sigma'(2) = 3$ and $\sigma'(3) = 1$ gives $\gamma_N(\sigma') = 10$. $\triangleleft$

Figure 6.2.3: The orders σ and σ' for the JiT sequencing situation of Example 6.2.1.

Naturally, an interesting question is how we can identify whether an order is optimal or not. Also, if we can find sufficient conditions for this, could we use these conditions to construct an optimal order? As it turns out, we can indeed find such conditions and use these to obtain an algorithm that constructs an optimal order for every sequencing situation in $JiT^{2,2}$.

First we focus on the sufficient conditions. For this, we introduce the following additional notation. Given a JiT sequencing situation $(N, \alpha, s, s_0) \in JiT^{2,2}$ and an order $\sigma \in \Pi(N)$, define the following classes of neighboring pairs:

$$\begin{aligned}
M^{hH}(\sigma) &= \{(i,j) \in (N \cup \{0\}) \times N \mid s_i = s^h,\ \alpha_j = \alpha^H,\ \sigma^{-1}(i) = \sigma^{-1}(j) - 1\},\\
M^{lL}(\sigma) &= \{(i,j) \in (N \cup \{0\}) \times N \mid s_i = s^l,\ \alpha_j = \alpha^L,\ \sigma^{-1}(i) = \sigma^{-1}(j) - 1\},\\
M^{hL}(\sigma) &= \{(i,j) \in (N \cup \{0\}) \times N \mid s_i = s^h,\ \alpha_j = \alpha^L,\ \sigma^{-1}(i) = \sigma^{-1}(j) - 1\},
\end{aligned}$$

and

$$M^{lH}(\sigma) = \{(i,j) \in (N \cup \{0\}) \times N \mid s_i = s^l,\ \alpha_j = \alpha^H,\ \sigma^{-1}(i) = \sigma^{-1}(j) - 1\}.$$

Note that the first superscript indicates the set-up time and the second superscript indicates the cost level. For every $\sigma \in \Pi(N)$, we have the following equalities:

$$|M^{hH}(\sigma)| + |M^{lH}(\sigma)| = |N^H|, \tag{6.1}$$

and

$$|M^{lL}(\sigma)| + |M^{hL}(\sigma)| = |N^L|. \tag{6.2}$$

If $s_0 = s^h$, we have for every order $\sigma \in \Pi(N)$ that

$$|M^{lH}(\sigma)| + |M^{lL}(\sigma)| = |N_l| - \mathbb{1}_{[s_{\sigma(|N|)}=s^l]}, \tag{6.3}$$

and

$$|M^{hH}(\sigma)| + |M^{hL}(\sigma)| = |N_h| + \mathbb{1}_{[s_{\sigma(|N|)}=s^l]}, \tag{6.4}$$

since $\mathbb{1}_{[s_{\sigma(|N|)}=s^l]} = 1 - \mathbb{1}_{[s_{\sigma(|N|)}=s^h]}$, $s_0 = s^h$ and the set-up time of the last player in the order does not incur costs for a player in N. If $s_0 = s^l$, we have

$$|M^{lH}(\sigma)| + |M^{lL}(\sigma)| = |N_l| + \mathbb{1}_{[s_{\sigma(|N|)}=s^h]}, \tag{6.5}$$

and

$$|M^{hH}(\sigma)| + |M^{hL}(\sigma)| = |N_h| - \mathbb{1}_{[s_{\sigma(|N|)}=s^h]}, \tag{6.6}$$

for every order $\sigma \in \Pi(N)$. Again, note that $\mathbb{1}_{[s_{\sigma(|N|)}=s^h]} = 1 - \mathbb{1}_{[s_{\sigma(|N|)}=s^l]}$. The following theorem states sufficient conditions for an order to be optimal. In general, an order satisfying these sufficient conditions need not exist. However, Proposition 6.2.5 shows that only in the specific case where $N_h^H \cup N_l^L = \emptyset$ such an order does not exists.

Theorem 6.2.2 Let $(N, \alpha, s, s_0) \in JiT^{2,2}$ and let $\sigma \in \Pi(N)$. If $s_{\sigma(|N|)} = \max_{i \in N} s_i$ and either $|M^{hH}(\sigma)| = 0$ or $|M^{lL}(\sigma)| = 0$, then σ is optimal.

Proof: Assume $s_0 = s^h$ and first consider the case where $\max_{i\in N} s_i = s^l$. This implies that $|N_h^H| = |N_h^L| = 0$. For $\sigma \in \Pi(N)$ we obtain

$$\gamma_N(\sigma) = \sum_{i\in N} s^l\alpha_i + (s_0 - s^l)\alpha_{\sigma(1)} = \sum_{i\in N} s^l\alpha_i + (s^h - s^l)\alpha_{\sigma(1)}.$$

Assume there exists an order $\sigma' \in \Pi(N)$ such that $|M^{hH}(\sigma')| = 0$, and take such a $\sigma' \in \Pi(N)$. As $s_0 = s^h$, we obtain that $\alpha_{\sigma'(1)} = \alpha^L$ and therefore $\sigma'(1) \in N_l^L$. Hence, for all $\sigma \in \Pi(N)$

$$\gamma_N(\sigma') \;=\; \sum_{i\in N} s^l\alpha_i + (s^h - s^l)\alpha^L \leq \gamma_N(\sigma),$$

so σ' is optimal.

Now assume there exists an order $\sigma'' \in \Pi(N)$ such that $|M^{lL}(\sigma'')| = 0$ and $|M^{hH}(\sigma'')| > 0$, and take such a $\sigma'' \in \Pi(N)$. Then either $|N_l^L| = 0$, which means that $N = N_l^H$ and every order is optimal, or $|N_l^L| = 1$ with $\sigma'(1) \in N_l^L$. In the last case $\gamma_N(\sigma'') = \sum_{i\in N} s^l\alpha_i + (s_0 - s^l)\alpha^L \leq \gamma_N(\sigma)$ for all $\sigma \in \Pi(N)$, and σ'' is optimal.

Now consider the case where $\max_{i\in N} s_i = s^h$. Take an arbitrary $\sigma \in \Pi(N)$. Take $B, D \in \mathbb{N}$ such that $B = |M^{hH}(\sigma)|$ and $D = |M^{lL}(\sigma)|$. Note that by Equation (6.2) and (6.4) we have

$$B - D = |M^{hH}(\sigma)| - |M^{lL}(\sigma)| = |N_h| - |N^L| + \mathbb{1}_{[s_{\sigma(|N|)}=s^l]}.$$

By (6.3) and (6.4) it holds that

$$\begin{aligned}
\gamma_N(\sigma) \;&=\; |M^{hH}(\sigma)|s^h\alpha^H + |M^{lH}(\sigma)|s^l\alpha^H + |M^{hL}(\sigma)|s^h\alpha^L + |M^{lL}(\sigma)|s^l\alpha^L \\
&=\; Bs^h\alpha^H + (|N_l| - \mathbb{1}_{[s_{\sigma(|N|)}=s^l]} - D)s^l\alpha^H \\
&\quad+\; (|N_h| + \mathbb{1}_{[s_{\sigma(|N|)}=s^l]} - B)s^h\alpha^L + Ds^l\alpha^L \\
&\geq\; (B - \min\{B,D\})s^h\alpha^H + (|N_l| - \mathbb{1}_{[s_{\sigma(|N|)}=s^l]} - D + \min\{B,D\})s^l\alpha^H \\
&\quad+\; (|N_h| + \mathbb{1}_{[s_{\sigma(|N|)}=s^l]} - B + \min\{B,D\})s^h\alpha^L + (D - \min\{B,D\})s^l\alpha^L \\
&=\; \max\{0, |N_h| - |N^L| + \mathbb{1}_{[s_{\sigma(|N|)}=s^l]}\}s^h\alpha^H \\
&\quad+\; \min\{|N_l| - \mathbb{1}_{[s_{\sigma(|N|)}=s^l]}, |N^H|\}s^l\alpha^H \\
&\quad+\; \min\{|N^L|, |N_h| + \mathbb{1}_{[s_{\sigma(|N|)}=s^l]}\}s^h\alpha^L \\
&\quad+\; \max\{|N^L| - |N_h| - \mathbb{1}_{[s_{\sigma(|N|)}=s^l]}, 0\}s^l\alpha^L \\
&\geq\; \max\{0, |N_h| - |N^L|\}s^h\alpha^H + \min\{|N_l|, |N^H|\}s^l\alpha^H \\
&\quad+\; \min\{|N^L|, |N_h|\}s^h\alpha^L + \max\{|N^L| - |N_h|, 0\}s^l\alpha^L,
\end{aligned}$$

where the first inequality follows from the observation that $(s^h - s^l)(\alpha^H - \alpha^L) > 0$. If $|N_h| - |N^L| \geq 0$, and therefore $|N_l| - |N^H| \leq 0$, then the second inequality follows from $(s^h - s^l)\alpha^H > 0$. If $|N_h| - |N^L| < 0$, then the second inequality follows from $(s^h - s^l)\alpha^L > 0$. The first inequality holds with equality if either $B = 0$ or $D = 0$, and the second inequality holds with equality if $s_{\sigma(|N|)} = s^h$. This shows that every order $\sigma \in \Pi(N)$ with $s_{\sigma(|N|)} = s^h$ and either $|M^{hH}(\sigma)| = 0$ or $|M^{lL}(\sigma)| = 0$ is optimal.

Next, assume $s_0 = s^l$ and first consider the case where $\max_{i \in N} s_i = s^l$. In this case, for any order $\sigma \in \Pi(N)$ we have

$$\gamma_N(\sigma) = \sum_{i \in N} s^l \alpha_i.$$

So, every order is optimal.

Now consider the case where $\max_{i \in N} s_i = s^h$. Take an arbitrary $\sigma \in \Pi(N)$. Take $B, D \in \mathbb{N}$ such that $B = |M^{hH}(\sigma)|$ and $D = |M^{lL}(\sigma)|$. Note that by Equation (6.2) and (6.6) we have

$$B - D = |M^{hH}(\sigma)| - |M^{lL}(\sigma)| = |N_h| - |N^L| - \mathbb{1}_{[s_{\sigma(|N|)} = s^h]}.$$

By (6.5) and (6.6) it holds that

$$\begin{aligned}
\gamma_N(\sigma) &= |M^{hH}(\sigma)|s^h\alpha^H + |M^{lH}(\sigma)|s^l\alpha^H + |M^{hL}(\sigma)|s^h\alpha^L + |M^{lL}(\sigma)|s^l\alpha^L \\
&= Bs^h\alpha^H + (|N_l| + \mathbb{1}_{[s_{\sigma(|N|)} = s^h]} - D)s^l\alpha^H \\
&+ (|N_h| - \mathbb{1}_{[s_{\sigma(|N|)} = s^h]} - B)s^h\alpha^L + Ds^l\alpha^L \\
&\geq (B - \min\{B, D\})s^h\alpha^H + (|N_l| + \mathbb{1}_{[s_{\sigma(|N|)} = s^h]} - D + \min\{B, D\})s^l\alpha^H \\
&+ (|N_h| - \mathbb{1}_{[s_{\sigma(|N|)} = s^h]} - B + \min\{B, D\})s^h\alpha^L + (D - \min\{B, D\})s^l\alpha^L \\
&= \max\{0, |N_h| - |N^L| - \mathbb{1}_{[s_{\sigma(|N|)} = s^h]}\}s^h\alpha^H \\
&+ \min\{|N_l| + \mathbb{1}_{[s_{\sigma(|N|)} = s^h]}, |N^H|\}s^l\alpha^H \\
&+ \min\{|N^L|, |N_h| - \mathbb{1}_{[s_{\sigma(|N|)} = s^h]}\}s^h\alpha^L \\
&+ \max\{|N^L| - |N_h| + \mathbb{1}_{[s_{\sigma(|N|)} = s^h]}, 0\}s^l\alpha^L \\
&\geq \max\{0, |N_h| - |N^L| - 1\}s^h\alpha^H + \min\{|N_l| + 1, |N^H|\}s^l\alpha^H \\
&+ \min\{|N^L|, |N_h| - 1\}s^h\alpha^L + \max\{|N^L| - |N_h| + 1, 0\}s^l\alpha^L,
\end{aligned}$$

where the first inequality follows from the observation that $(s^h - s^l)(\alpha^H - \alpha^L) > 0$.

If $|N_h| - |N^L| \leq 0$, then the second inequality follows from $(s^l - s^h)\alpha^L < 0$. If $|N_h| - |N^L| > 0$, then the second inequality follows from $(s^l - s^h)\alpha^H < 0$. The first inequality holds with equality if either $B = 0$ or $D = 0$, and the second inequality holds with equality if $s_{\sigma(|N|)} = s^h$.
Hence for both values of σ_0 every order $\sigma \in \Pi(N)$ with $s_{\sigma(|N|)} = s^h$ and either $|M^{hH}(\sigma)| = 0$ or $|M^{lL}(\sigma)| = 0$ is optimal. $\square$

The optimality conditions in Theorem 6.2.2 consist of two parts: the first condition states that it is optimal to place a player with highest set-up time possible at the last position. The second condition means that it is optimal to place players with low costs behind players with high set-up time and players with high costs behind players with low set-up time. These conditions are used in the following algorithm. The first condition is explicitly taken care of in step 2, the second condition is dealt with in step 3. Step 4 deals with these optimality conditions more implicitly, which is demonstrated in Example 6.2.3.

Algorithm 1

Input: a sequencing situation $(N, \alpha, s, s_0) \in JiT^{2,2}$.
Output: an order $\tilde{\sigma} \in \Pi(N)$.
Step 1. Initialize $p = 1$ and $C_1^1 = N$.
Step 2. Define

$$C_p^2 = \begin{cases} C_p^1 \backslash N_h & \text{if } |C_p^1 \cap N_h| = 1 \text{ and } p \neq |N|; \\ C_p^1 & \text{else.} \end{cases}$$

Step 3. Define

$$C_p^3 = \begin{cases} C_p^2 \cap N^L & \text{if } s_{\tilde{\sigma}(p-1)} = s^h \text{ and } C_p^2 \cap N^L \neq \emptyset; \\ C_p^2 \cap N^H & \text{if } s_{\tilde{\sigma}(p-1)} = s^l \text{ and } C_p^2 \cap N^H \neq \emptyset; \\ C_p^2 & \text{else.} \end{cases}$$

Step 4. Define

$$C_p^4 = \begin{cases} C_p^3 \cap N_l & \text{if } C_p^3 \cap N_l \neq \emptyset; \\ C_p^3 & \text{else.} \end{cases}$$

Step 5. Choose a job $i \in C_p^4$ and define $\tilde{\sigma}(p) = i$.
Step 6. If $p = |N|$, stop.

If $p < |N|$, set $p = p + 1$ and, subsequently, set $C_p^1 = C_{p-1}^1 \backslash \{\tilde{\sigma}(p-1)\}$. Next, return to step 2.

The notation $\tilde{\sigma}$ is used for an order provided by the algorithm. The algorithm generates this order by filling up all positions in the order from front to back. For every position, the set of candidate players is narrowed down in a few steps. In the algorithm, $C_p^4 \subseteq C_p^3 \subseteq C_p^2 \subseteq C_p^1 \in 2^N$ are the sets of candidate players for the p^{th} position in this order.

Roughly speaking, the algorithm puts the jobs in an alternating sequence, that is in a way that high set-up time meets low cost of spending a unit of time in the system and vice versa. Hereby it takes into account that a job with high set-up time should be left over for the last position in the sequence.

Example 6.2.3 Consider the JiT sequencing situation $\Psi = (N, \alpha, s, s_0)$, where we have $\alpha = (2, 2, 1, 1)$, $s = (3, 1, 3, 1)$ and $s_0 = 3$. We have $N_h^H = \{1\}$, $N_l^H = \{2\}$, $N_h^L = \{3\}$, and $N_l^L = \{4\}$. As $|N_h| = 2$, we have $C_1^2 = C_1^1 = N$ (see Table 6.2.2). In step 3 we obtain $C_1^3 = \{3, 4\}$ as $s_0 = s^h$. Step 4 further narrows down the set of candidate players for the first position, as $C_1^4 = \{4\}$. Therefore, we obtain $\tilde{\sigma}(1) = 4$. Now that player 4 is placed, we have $C_2^2 = C_2^1 = \{1, 2, 3\}$. In step 3 of iteration 2, we obtain $C_2^3 = \{1, 2\}$ and in step 4 we obtain $C_2^4 = \{2\}$ so $\tilde{\sigma}(2) = 2$. In the third iteration, $C_3^2 = C_3^1 = \{1, 3\}$ and $C_3^4 = C_3^3 = \{1\}$ so $\tilde{\sigma}(3) = 1$ and $\tilde{\sigma}(4) = 3$. It is

p	1	2	3	4
C_p^1	N	$\{1,2,3\}$	$\{1,3\}$	$\{3\}$
C_p^2	N	$\{1,2,3\}$	$\{1,3\}$	$\{3\}$
C_p^3	$\{3,4\}$	$\{1,2\}$	$\{1\}$	$\{3\}$
C_p^4	$\{4\}$	$\{2\}$	$\{1\}$	$\{3\}$
$\tilde{\sigma}(p)$	4	2	1	3

Table 6.2.2: Sets of candidate players in Example 6.2.3

easily seen that a player with high set-up time is placed last. Furthermore, players with high costs are placed behind players with low set-up time and the other way around (see Figure 6.2.4). We obtain $\gamma_N(\tilde{\sigma}) = 10$ which is indeed optimal. Also, note the importance of step 4 of the algorithm: if we would place an arbitrary player in C_2^3 at position 2, we could have ended up with the order σ' such that $\sigma'(1) = 4$, $\sigma'(2) = 1$, $\sigma'(3) = 2$, $\sigma'(4) = 3$, with $\gamma_N(\sigma') = 12$. $\triangleleft$

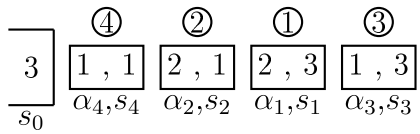

Figure 6.2.4: Order $\tilde{\sigma}$ provided by Algorithm 1 in Example 6.2.3

Now we are ready to prove that Algorithm 1 provides an optimal order, for every sequencing situation $(N, \alpha, s, s_0) \in JiT^{2,2}$.

Theorem 6.2.4 Let $\Psi = (N, \alpha, s, s_0) \in JiT^{2,2}$. Then Algorithm 1 provides an optimal order $\tilde{\sigma}$ for N.

Proof: Let $\tilde{\sigma} \in \Pi(N)$ be an order provided by Algorithm 1. In Step 2 of the algorithm, it is made sure that there is always a player with the highest available set-up time left to place at the last position. Hence, $s_{\tilde{\sigma}(|N|)} = s^h$, unless $N_h^H \cup N_h^L = \emptyset$ which implies that there is in fact only one value for s_i and $s_{\tilde{\sigma}(|N|)} = s^l = \max_{i \in N} s_i$. We prove that either optimality of $\tilde{\sigma}$ follows directly from Theorem 6.2.2, i.e., $|M^{hH}(\tilde{\sigma})| = 0$ or $|M^{lL}(\tilde{\sigma})| = 0$, or $N_h^H \cup N_l^L = \emptyset$. For the latter case we show that $\tilde{\sigma}$ is optimal as well.
Assume that optimality of $\tilde{\sigma}$ does not follow directly from Theorem 6.2.2, i.e., $|M^{hH}(\tilde{\sigma})| > 0$ and $|M^{lL}(\tilde{\sigma})| > 0$. Then there exist $p, r \in \{0, ..., |N| - 1\}$ such that $(\tilde{\sigma}(p), \tilde{\sigma}(p+1)) \in M^{hH}(\tilde{\sigma})$ and $(\tilde{\sigma}(r), \tilde{\sigma}(r+1)) \in M^{lL}(\tilde{\sigma})$.
Assume $r < p$. According to the algorithm, job $\tilde{\sigma}(r+1)$ is only placed behind job $\tilde{\sigma}(r)$ if there is no job j with $\alpha_j = \alpha^H$ left that is not yet placed, or there is only one job j with $\alpha_j = \alpha^H$ left, but this job has to be reserved for the last spot because it is the only remaining job with high set-up time. In the first case, we have a contradiction, since job $\tilde{\sigma}(p+1)$ is not yet placed. The second case also results in a contradiction, since both $s_{\tilde{\sigma}(p)} = s^h$ and $s_{\tilde{\sigma}(|N|)} = s^h$.
Now assume $p < r$. According to the algorithm, job $\tilde{\sigma}(p+1)$ is only placed behind job $\tilde{\sigma}(p)$ if there is no job j with $\alpha_j = \alpha^L$ left that is not yet placed, or there is only one job j with $\alpha_j = \alpha^L$ left, but this job has to be reserved for the last spot because it is the only remaining job with high set-up time. In the first case, we have a contradiction, since job $\tilde{\sigma}(r+1)$ is not yet placed.
The second case can only hold if $r + 1 = |N|$. For all jobs $i \in \{\tilde{\sigma}(p+1), ..., \tilde{\sigma}(r)\}$ it then must hold that $s_i = s^l$, otherwise job $\tilde{\sigma}(r+1)$ would have been placed at

position $p+1$. Furthermore, $\alpha_i = \alpha^H$ otherwise job i would have been placed at position $p+1$ as this would avoid the combination of s^h and α^H. So, we obtain that $i \in N_l^H$ for all $i \in \{\tilde{\sigma}(p+1), ..., \tilde{\sigma}(r)\}$. Since the algorithm first places the jobs in N_l^H before placing the jobs in N_h^H, and $\tilde{\sigma}(|N|) \notin N_h^H$, we obtain that $N_h^H = \emptyset$ and therefore $\tilde{\sigma}(p) \in N_h^L$. Furthermore, if there existed a job $i \in N_l^L$ then the algorithm would place every job in N_l^H directly behind this job. But since $\tilde{\sigma}(p) \notin N_l^L$ this implies that $N_l^L = \emptyset$. Hence, the second case only allows players in N_l^H and N_h^L, so $N_h^H \cup N_l^L = \emptyset$. If $s_0 = s^l$, then the algorithm first places all players in N_l^H and then all players in N_h^L, which contradicts $\tilde{\sigma}(s) \in N_h^L$. So, $s_0 = s^h$ and the solution provided by the algorithm for this situation (first all players in N_h^L but one, then all players in N_l^H and finally the last player in N_h^L) is clearly optimal. □

The proof of Theorem 6.2.4 implies the following.

Proposition 6.2.5 *If $N_h^H \cup N_l^L \neq \emptyset$, then every order provided by Algorithm 1 satisfies the sufficient conditions of Theorem 6.2.2.*

6.3 JiT sequencing games

In the previous section we addressed the problem of finding an optimal order for $JiT^{2,2}$ sequencing situations. An additional question is how the total costs of such an optimal order should be allocated among the players. To answer this question, we will use the framework of TU-games. For the grand coalition we employ a worst case approach and consider the players in N to be part of a larger system with players in the system before and after the players in N. The worst case approach comprises that we assume that the system is left behind with high set-up time by the players outside the grand coalition. We provide a game theoretic analysis of those instances of $JiT^{2,2}$ where $s_0 = s^h$, denoted by $JiT_h^{2,2}$. We assume that by cooperating, every coalition $S \in \mathcal{N}$ can form any order $\sigma \in \Pi(S)$ at the first $|S|$ spots in the sequence. Thus we employ a pessimistic view for both the grand coalition and for subcoalitions, in the sense that the set-up time for the first player in the order σ equals $s_0 = s^h$. This setup allows us to measure the value of every coalition consistently over all coalitions, and independent of the players outside the coalition.

Let $\Psi = (N, \alpha, s, s_0)$ be a JiT sequencing situation. We will define the costs for coalition $S \in \mathcal{N}$ as the costs of an optimal order in the JiT sequencing problem

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$
$v^{\Psi}(S)$	0	0	0	0	4	0	4	2

S	$\{2,4\}$	$\{3,4\}$	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	N
$v^{\Psi}(S)$	4	2	4	8	4	6	8

Table 6.3.1: JiT sequencing game (N, v^{Ψ}) of Example 6.3.1

(S, α', s', s'_0), where $\alpha' \in \mathbb{R}^S$ and $s' \in \mathbb{R}^{S\cup\{0\}}$ are such that $\alpha'_i = \alpha_i$ for all $i \in S$, and $s'_i = s_i$ for all $i \in S \cup \{0\}$. Given a JiT sequencing situation $\Psi = (N, \alpha, s, s_0)$ and a coalition $S \in \mathcal{N}$, we denote by σ^*_S an optimal order of the situation (S, α', s', s'_0). Hence, formally, the *JiT sequencing game* (N, v^{Ψ}) is defined by

$$v^{\Psi}(S) = \sum_{i\in S} \gamma_i(\sigma^*_{\{i\}}) - \gamma_S(\sigma^*_S),$$

for all $S \in \mathcal{N}$. Clearly, $v^{\Psi}(\{i\}) = 0$ for every $i \in N$.

Example 6.3.1 Reconsider the JiT sequencing situation of Example 6.2.3, where $\alpha = (2, 2, 1, 1)$, $s = (3, 1, 3, 1)$ and $s_0 = 3$. It is seen that $\gamma_1(\sigma^*_{\{1\}}) = \gamma_2(\sigma^*_{\{2\}}) = 6$ and $\gamma_3(\sigma^*_{\{3\}}) = \gamma_4(\sigma^*_{\{4\}}) = 3$. Take $S = \{1, 2, 4\}$. The optimal order σ^*_S is such that $\sigma^*_S(1) = 4$, $\sigma^*_S(2) = 2$, and $\sigma^*_S(3) = 1$, which results in total costs $\gamma_S(\sigma^*_S) = 7$. Hence, we have $v^{\Psi}(S) = (6 + 6 + 3) - 7 = 8$. The JiT sequencing game (N, v^{Ψ}) is given by Table 6.3.1. Note that (N, v^{Ψ}) is not convex, since

$$v^{\Psi}(\{3, 4\}) - v^{\Psi}(\{4\}) = 2 > 0 = v^{\Psi}(N) - v^{\Psi}(\{1, 2, 4\}).$$

$\triangleleft$

For $\Psi = (N, \alpha, s, s_0) \in JiT^{2,2}_h$ we can explicitly express the value of each coalition in terms of the number of players in the different player classes in the JiT sequencing situation.

Lemma 6.3.2 *Let* $\Psi = (N, \alpha, s, s_0) \in JiT^{2,2}_h$ *. Then*

$$v^{\Psi}(S) = \begin{cases} (|S_l^H| + |S_l^L|)(s^h - s^l)\alpha^H & \text{if } S_h^H \neq \emptyset \text{ and } |S_h^H| \geq |S_l^L|; \\ |S_h^H|(s^h - s^l)(\alpha^H - \alpha^L) + |S_l^H|(s^h - s^l)\alpha^H & \text{if } S_h^H \neq \emptyset \text{ and } |S_h^H| < |S_l^L|; \\ \quad + |S_l^L|(s^h - s^l)\alpha^L & \\ |S_l^H|(s^h - s^l)\alpha^H + |S_l^L|(s^h - s^l)\alpha^L & \text{if } S_h^H = \emptyset, S_l^L \neq \emptyset \\ & \text{and } S_h^L \neq \emptyset; \\ |S_l^H|(s^h - s^l)\alpha^H + (|S_l^L| - 1)(s^h - s^l)\alpha^L & \text{if } S_h^H = \emptyset, S_l^L \neq \emptyset \\ & \text{and } S_h^L = \emptyset; \\ (|S_l^H| - 1)(s^h - s^l)\alpha^H + (s^h - s^l)\alpha^L & \text{if } S_h^H = \emptyset, S_l^L = \emptyset, S_l^H \neq \emptyset \\ & \text{and } S_h^L \neq \emptyset; \\ (|S_l^H| - 1)(s^h - s^l)\alpha^H & \text{if } S_h^H = \emptyset, S_l^L = \emptyset \\ & \text{and } S_h^L = \emptyset; \\ 0 & \text{if } S_h^H = \emptyset, S_l^L = \emptyset \\ & \text{and } S_l^H = \emptyset, \end{cases}$$

for all $S \in \mathcal{N}$.

Proof: Here, we will only prove the first case. As the other cases follow from a similar reasoning, we refer to the appendix for the proof of those cases. First of all, we have

$$\sum_{i \in S} \gamma_i(\sigma^*_{\{i\}}) = (|S_h^H| + |S_l^H|)s^h\alpha^H + (|S_h^L| + |S_l^L|)s^h\alpha^L,$$

for every $S \in \mathcal{N}$.

Take $S \in \mathcal{N}$ such that $S_h^H \neq \emptyset$ and $|S_h^H| \geq |S_l^L|$. Since $S_h^H \neq \emptyset$, it holds for every (optimal) order $\tilde{\sigma}_S$ provided by Algorithm 1 that $s_{\sigma^*_S(|S|)} = s^h$. Furthermore, since $S_h^H \neq \emptyset$ we have by Proposition 6.2.5 that either $|M^{hH}(\tilde{\sigma}_S)| = 0$ or $|M^{lL}(\tilde{\sigma}_S)| = 0$. It must hold that $|M^{lL}(\tilde{\sigma}_S)| = 0$, since $|S_h^H| \geq |S_l^L|$ together with (6.1) and (6.3) implies that $|M^{hH}(\tilde{\sigma}_S)| \geq |M^{lL}(\tilde{\sigma}_S)|$. So, we have

$$\begin{aligned} \gamma_S(\tilde{\sigma}_S) &= |M^{hH}(\tilde{\sigma}_S)|s^h\alpha^H + |M^{lH}(\tilde{\sigma}_S)|s^l\alpha^H + |M^{hL}(\tilde{\sigma}_S)|s^h\alpha^L + |M^{lL}(\tilde{\sigma}_S)|s^l\alpha^L \\ &= (|S_h^H| - |S_l^L|)s^h\alpha^H + (|S_l^H| + |S_l^L|)s^l\alpha^H + (|S_h^L| + |S_l^L|)s^h\alpha^L, \end{aligned}$$

and we may conclude that

$$\begin{aligned}
v^{\Psi}(S) &= \sum_{i\in S}\gamma_i(\tilde{\sigma}_{\{i\}}) - \gamma_S(\tilde{\sigma}_S) \\
&= (|S_h^H| + |S_l^H|)s^h\alpha^H + (|S_h^L| + |S_l^L|)s^h\alpha^L, \\
&- \big((|S_h^H| - |S_l^L|)s^h\alpha^H + (|S_l^H| + |S_l^L|)s^l\alpha^H + (|S_h^L| + |S_l^L|)s^h\alpha^L\big) \\
&= (|S_l^H| + |S_l^L|)(s^h - s^l)\alpha^H.
\end{aligned}$$

□

All marginal contributions can be readily determined from Lemma 6.3.2.

Corollary 6.3.3 *Let $\Psi = (N, \alpha, s, s_0) \in JiT_h^{2,2}$, $i \in N$ and $S \in 2^{N\setminus\{i\}}$. If $i \in N_h^H$,*

$$v^{\Psi}(S\cup\{i\}) - v^{\Psi}(S) = \begin{cases} (s^h - s^l)\alpha^H & \text{if } S \neq \emptyset, S_h^H = \emptyset \text{ and } S_h^L = \emptyset; \\ (s^h - s^l)(\alpha^H - \alpha^L) & \text{if } S_h^H \neq \emptyset \text{ and } |S_h^H| < |S_l^L|, \\ & \text{or if } S_h^H = \emptyset, S_l^L \neq \emptyset \text{ and } S_h^L \neq \emptyset, \\ & \text{or if } S_h^H = \emptyset, S_l^L = \emptyset, S_l^H \neq \emptyset \\ & \quad \text{and } S_h^L \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

If $i \in N_l^L$,

$$v^{\Psi}(S\cup\{i\}) - v^{\Psi}(S) = \begin{cases} 0 & \text{if } S = \emptyset; \\ (s^h - s^l)\alpha^H & \text{if } S_h^H \neq \emptyset \text{ and } |S_h^H| > |S_l^L|, \\ & \text{or if } S_h^H = \emptyset, S_l^L = \emptyset \text{ and } S_l^H \neq \emptyset; \\ (s^h - s^l)\alpha^L & \text{otherwise.} \end{cases}$$

If $i \in N_l^H$,

$$v^{\Psi}(S\cup\{i\}) - v^{\Psi}(S) = \begin{cases} 0 & \text{if } S = \emptyset; \\ (s^h - s^l)\alpha^L & \text{if } S_h^H = \emptyset, S_l^L = \emptyset, S_l^H = \emptyset \text{ and } S_h^L \neq \emptyset; \\ (s^h - s^l)\alpha^H & \text{otherwise.} \end{cases}$$

Finally, if $i \in N_h^L$,

$$v^{\Psi}(S\cup\{i\}) - v^{\Psi}(S) = \begin{cases} (s^h - s^l)\alpha^L & \text{if } S \neq \emptyset, S_h^H = \emptyset \text{ and } S_h^L = \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

The following example illustrates both Lemma 6.3.2 and Corollary 6.3.3.

Example 6.3.4 Consider the JiT sequencing situation $\Psi = (N, \alpha, s, s_0) \in JiT_h^{2,2}$ such that $N = \{1,2,3,4,5,6\}$, $\alpha = (5,5,5,2,2,2)$, $s = (3,3,1,3,1,1)$ and $s_0 = 3$. This means that $N_h^H = \{1,2\}$, $N_l^H = \{3\}$, $N_h^L = \{4\}$ and $N_l^L = \{5,6\}$. Consider the coalition $S = \{1,3,6\}$. The stand-alone costs for the players in this coalition are $\gamma_1(\tilde{\sigma}_{\{1\}}) = \gamma_3(\tilde{\sigma}_{\{3\}}) = 15$ and $\gamma_6(\tilde{\sigma}_{\{6\}}) = 6$. Using Algorithm 1, we obtain $\tilde{\sigma}_S(1) = 6$, $\tilde{\sigma}_S(2) = 3$ and $\tilde{\sigma}_S(3) = 2$ for the optimal order $\tilde{\sigma}_S$ for coalition S. Hence, $\gamma_S(\tilde{\sigma}_S) = 16$ and $v^\Psi(S) = \sum_{i \in S} \gamma_i(\tilde{\sigma}_{\{i\}}) - \gamma_S(\tilde{\sigma}_S) = 20$. This is in line with Lemma 6.3.2, as $(|S_l^H| + |S_l^L|)(s^h \alpha^H - s^l \alpha^H) = 2(15-5) = 20$. By Corollary 6.3.3, we have $v(\{2,3,6\}) - v(\{3,6\}) = (s^h - s^l)\alpha^H = 10$ which is easily verified, as $v(\{3,6\}) = 10$. On the other hand, the marginal contribution of player 2 when he joins coalition $\{1,3,6\}$ equals zero, as $v(\{1,2,3,6\}) = v(\{1,3,6\})$. $\triangleleft$

We use the expressions from Lemma 6.3.2 to show that every JiT sequencing game has a nonempty core. To this end, we define the *large instance based allocation rule* θ on the class of JiT sequencing situations $JiT_h^{2,2}$.

Definition 6.3.5 Let $\Psi = (N, \alpha, s, s_0) \in JiT_h^{2,2}$. Then, for all $i \in N$,

$$\theta_i(\Psi) = (s^h - s_i)\alpha_i + \begin{cases} \frac{(\alpha^H - \alpha^L)(s^h - s^l)}{2} & \text{if } i \in N_h^H \cup N_l^L \\ & \quad \text{and } |N_h^H| = |N_l^L|; \\ (\alpha^H - \alpha^L)(s^h - s^l) & \text{if } i \in N_h^H \text{ and } |N_h^H| < |N_l^L|, \\ & \text{or if } i \in N_l^L \\ & \quad \text{and } |N_h^H| > |N_l^L|; \\ -\frac{1}{|N_l^L|}(s^h - s^l)\alpha^L & \text{if } i \in N_l^L \text{ and } |N_h| = 0; \\ -\frac{1}{|N_l^H|}(\alpha^H - \alpha^L)(s^h - s^l) & \text{if } i \in N_l^H, |N_h^L| > 0 \text{ and} \\ & \quad |N_h^H| = |N_l^L| = 0; \\ -\frac{1}{|N_l^H|}(s^h - s^l)\alpha^H & \text{if } i \in N_l^H \text{ and } N = N_l^H; \\ 0 & \text{otherwise.} \end{cases}$$

The common part of the expression for $\theta(\Psi)$, $(s^h - s_i)\alpha_i$, gives an estimation of the cost savings that can be attributed to player i. This estimation is based on the marginal costs of player i entering in a fictive, 'large' coalition. The part $s^h\alpha_i$ are the stand-alone costs of player i. Now assume there is an order $\sigma \in \Pi(N)$

where $s_{\sigma(k)} = s_i$ for some k. If player i is placed in between player $\sigma(k)$ and player $\sigma(k+1)$, then the marginal costs equal $s_i\alpha_i$. Hence, we estimate the cost savings by $(s^h - s_i)\alpha_i$.

The second part serves as a correction to this estimation: a player in N_h^H and a player in N_l^L together are responsible for more cost savings than we already allocated to them. These additional cost savings go to the minority, the players in N_h^H if $|N_h^H| < |N_l^L|$ and the players in N_l^L if $|N_l^L| < |N_h^H|$, and is shared equally if $|N_h^H| = |N_l^L|$. The other corrections are due to boundary cases of the player set: for example, if there are no players in both N_h^H and N_l^L, then the cost savings attributed to players in N_l^H is overestimated, and is corrected.

Example 6.3.6 Reconsider the JiT sequencing situation Ψ of Example 6.3.4. We have $|N_h^H| = |N_l^L| > 0$, so $\theta(\Psi) = (3, 3, 10, 0, 7, 7)$. $\triangleleft$

For every JiT sequencing situation $\Psi \in JiT_h^{2,2}$, the large instance based allocation rule provides a core element for the corresponding game (N, v^Ψ).

Theorem 6.3.7 Let $\Psi = (N, \alpha, s, s_0) \in JiT_h^{2,2}$. Then $\theta(\Psi) \in C(v^\Psi)$.

Proof: We consider four different cases, and use Lemma 6.3.2 and Definition 6.3.5 in each of these cases.

(i) Assume $|N_h^H| = 0$, $|N_l^L| = 0$ and $|N_h^L| > 0$. For $S \in \mathcal{N}$ we have

$$\begin{aligned}
\theta_S(\Psi) - v^\Psi(S) &\geq \sum_{i \in S_l^H} \left((s^h - s^l)\alpha^H - \frac{1}{|N_l^H|}(s^h - s^l)(\alpha^H - \alpha^L) \right) \\
&- (|S_l^H| - 1)(s^h - s^l)\alpha^H - (s^h - s^l)\alpha^L \\
&= |S_l^H| \left((s^h - s^l)\alpha^H - \frac{1}{|N_l^H|}(s^h - s^l)(\alpha^H - \alpha^L) \right) \\
&- (|S_l^H| - 1)(s^h - s^l)\alpha^H - (s^h - s^l)\alpha^L \\
&= (1 - \frac{|S_l^H|}{|N_l^H|})(s^h - s^l)(\alpha^H - \alpha^L) \\
&\geq 0,
\end{aligned}$$

with equality if $S = N$, and therefore $\theta(\Psi) \in C(v^\Psi)$.

(ii) Assume $|N_h^H| + |N_l^L| > 0$ and $|N_h| > 0$. For $S \in \mathcal{N}$ we have

$$\begin{aligned}
\theta_S(\Psi) - v^\Psi(S) &\geq |S_l^H|(s^h - s^l)\alpha^H + |S_l^L|(s^h - s^l)\alpha^L \\
&+ \min\{|S_h^H|, |S_l^L|\}(s^h - s^l)(\alpha^H - \alpha^L) \\
&- |S_l^H|(s^h - s^l)\alpha^H - |S_l^L|(s^h - s^l)\alpha^L \\
&- \min\{|S_h^H|, |S_l^L|\}(s^h - s^l)(\alpha^H - \alpha^L) \\
&\geq 0,
\end{aligned}$$

again with equality if $S = N$, and therefore $\theta(\Psi) \in C(v^\Psi)$.

(iii) Assume $|N_h| = 0$ and $|N_l^L| > 0$. For $S \in \mathcal{N}$ we have

$$\begin{aligned}
\theta_S(\Psi) - v^\Psi(S) &\geq |S_l^H|(s^h - s^l)\alpha^H + |S_l^L|((s^h - s^l)\alpha^L - \frac{1}{|N_l^L|}(s^h - s^l)\alpha^L) \\
&- |S_l^H|(s^h - s^l)\alpha^H - (|S_l^L| - 1)(s^h - s^l)\alpha^L \\
&= (1 - \frac{|S_l^L|}{|N_l^L|})(s^h - s^l)\alpha^L \\
&\geq 0,
\end{aligned}$$

with equality if $S = N$, and therefore $\theta(\Psi) \in C(v^\Psi)$.

(iv) Finally, assume $|N_h^H| = |N_l^L| = |N_h^L| = 0$. For $S \in \mathcal{N}$ we have

$$\begin{aligned}
\theta_S(\Psi) - v^\Psi(S) &= |S_l^H|((s^h - s^l)\alpha^H - \frac{1}{|N_l^H|}(s^h - s^l)\alpha^H) \\
&- (|S_l^H| - 1)(s^h - s^l)\alpha^H \\
&= (1 - \frac{|S_l^H|}{|N_l^H|})(s^h - s^l)\alpha^H \\
&\geq 0,
\end{aligned}$$

with equality if $S = N$, and therefore $\theta(\Psi) \in C(v^\Psi)$.

□

For every JiT sequencing situation $\Psi = (N, \alpha, s, s_0) \in JiT_h^{2,2}$ such that $|N_h^H| = |N_l^L| > 1$, the core is the convex hull of two vectors. We show this in Theorem 6.3.8. This structure of the core is similar to the one for a specific type of assignment games, called Böhm-Bawerk horse market games (Böhm-Bawerk (1891), see also Chapter 5 in this dissertation), in the sense that the core consists of a line segment, where one extreme point is 'buyer'-optimal and the other extreme point is 'seller'-optimal. In our setting, the players in N_h^H act as the buyers and players in N_l^L act as the sellers. Translated into the terminology of horse market games, every player in N_h^H is interested to buy the right on low set-up time, and every player in N_l^L is interested to sell the right on low set-up time. By interacting on the market, every pair of a buyer and a seller creates a profit that can be shared in an arbitrary, nonnegative way. This profit equals $(s^h - s^l)\alpha^H$.

For a JiT sequencing situation $\Psi = (N, \alpha, s, s_0) \in JiT_h^{2,2}$ with $|N_h^H| = |N_l^L| > 1$, and for $j \in N$ define

$$\overline{\theta}_j(\Psi) = (s^h - s_j)\alpha_j + \begin{cases} (s^h - s^l)(\alpha^H - \alpha^L) & \text{if } j \in N_h^H; \\ 0 & \text{else,} \end{cases}$$

$$\underline{\theta}_j(\Psi) = (s^h - s_j)\alpha_j + \begin{cases} (s^h - s^l)(\alpha^H - \alpha^L) & \text{if } j \in N_l^L; \\ 0 & \text{else.} \end{cases}$$

Note that $\overline{\theta}(\Psi)$ corresponds to the 'buyer'-optimal allocation, and $\underline{\theta}(\Psi)$ to the 'seller'-optimal allocation.

Theorem 6.3.8 Let $\Psi = (N, \alpha, s, s_0) \in JiT_h^{2,2}$ be such that $|N_h^H| = |N_l^L| > 1$. Then $C(v^\Psi) = Conv\{\overline{\theta}(\Psi), \underline{\theta}(\Psi)\}$ and

$$\theta(\Psi) = \frac{1}{2}(\overline{\theta}(\Psi) + \underline{\theta}(\Psi)),$$

which coincides the barycenter of the core.

Proof: First we establish that $Conv\{\overline{\theta}(\Psi), \underline{\theta}(\Psi)\} \subseteq C(v^\Psi)$, by showing that $\overline{\theta}(\Psi) \in C(v^\Psi)$ and $\underline{\theta}(\Psi) \in C(v^\Psi)$.
Let $S \in \mathcal{N}$. Then we have that

$$\begin{aligned}
\sum_{i \in S} \overline{\theta}_i(\Psi) - v^{\Psi}(S) &= \sum_{i \in S} (s^h - s_i)\alpha_i + |S_h^H|(s^h - s^l)(\alpha^H - \alpha^L) - v^{\Psi}(S) \\
&\geq |S_l^H|(s^h - s^l)\alpha^H + |S_l^L|(s^h - s^l)\alpha^L + |S_h^H|(s^h - s^l)(\alpha^H - \alpha^L) \\
&- \left(|S_l^H|(s^h - s^l)\alpha^H + |S_l^L|(s^h - s^l)\alpha^L\right) \\
&+ \min\{|S_h^H|, |S_l^L|\}(s^h - s^l)(\alpha^H - \alpha^L) \\
&= \left(|S_h^H| - \min\{|S_h^H|, |S_l^L|\}\right)(s^h - s^l)(\alpha^H - \alpha^L) \\
&\geq 0.
\end{aligned}$$

Note that the first inequality follows from Lemma 6.3.2, and that for $S = N$ the two inequalities hold with equality. Hence, $\overline{\theta}(\Psi) \in C(v^{\Psi})$. A similar reasoning shows that $\overline{\theta}(\Psi) \in C(v^{\Psi})$.

Now we show that $C(v^{\Psi}) \subseteq Conv\{\overline{\theta}(\Psi), \underline{\theta}(\Psi)\}$. It suffices to show that for every $x \in C(v^{\Psi})$ it holds that:

(i) $x_i \geq 0$ for every $i \in N$.

(ii) $x_i = 0$ for every $i \in N_h^L$.

(iii) $x_i = (s^h - s^l)\alpha^H$ for every $i \in N_l^H$.

(iv) $x_i + x_j = (s^h - s^l)\alpha^H$ for every $i \in N_h^H$ and $j \in N_l^L$ and, therefore, $x_i = x_k$ for all $i, k \in N_h^H$, and $x_j = x_r$ for all $j, r \in N_l^L$.

(v) $(s^h - s^l)\alpha^L \leq x_i \leq (s^h - s^l)\alpha^H$ for every $i \in N_l^L$.

Note that (iv) and (v) together imply $0 \leq x_i \leq (s^h - s^l)(\alpha^H - \alpha^L)$ for every $i \in N_h^H$. We prove (i) - (v) point by point. Take $x \in C(v^{\Psi})$.

(i) As $x_i \geq v(\{i\})$ and $v(\{i\}) = 0$ for every $i \in N$, we have $x_i \geq 0$ for every $i \in N$.

(ii) As $v^{\Psi}(N \backslash N_h^L) = v^{\Psi}(N)$, we obtain $x_i = 0$ for all $i \in N_h^L$.

(iii) Take $i \in N_l^H$, and take $j \in N_h^H$, $k \in N_l^L$. Then

$$\begin{aligned}
\sum_{r\in N} x_r + x_i &= \sum_{r\in N_h^H\cup\{i\}} x_r + \sum_{r\in N\backslash N_h^H} x_r \\
&\geq v^\Psi(\{i,j,k\}) + v^\Psi(N\backslash\{j,k\}) \\
&= 2(s^h - s^l)\alpha^H + (|N_l^H| + |N_l^L|)(s^h - s^l)\alpha^H \\
&= v^\Psi(N) + (s^h - s^l)\alpha^H \\
&= \sum_{r\in N} x_r + (s^h - s^l)\alpha^H.
\end{aligned}$$

The second equality holds by the observation that $N_h^H \cap (N\backslash\{j,k\}) \neq \emptyset$ and $N_l^L \cap (N\backslash\{j,k\}) \neq \emptyset$, which means that $N\backslash\{j,k\}$ is contained in the first case of Lemma 6.3.2. We obtain $x_i \geq (s^h - s^l)\alpha^H$ and by Corollary 6.3.3, we have $v^\Psi(N) - v^\Psi(N\backslash\{i\}) = (s^h - s^l)\alpha^H$. Therefore, $x_i = (s^h - s^l)\alpha^H$.

(iv) Take $i \in N_h^H$ and $j \in N_l^L$. As $v^\Psi(\{i,j\}) = (s^h - s^l)\alpha^H = v^\Psi(N) - v^\Psi(N\backslash\{i,j\})$ we have $x_i + x_j = (s^h - s^l)\alpha^H$.

(v) Take $j \in N_h^H$ and $i, k \in N_l^L$ with $j \neq k$. The observation that $v(\{i,j,k\}) = (s^h - s^l)(\alpha^H - \alpha^L) + 2(s^h - s^l)\alpha^L$ and $v(N\backslash\{j,k\}) = (|N_l^H| + |N_l^L| - 1)(s^h - s^l)\alpha^H$ leads with a similar reason as with (iii) to $x_i \geq (s^h - s^l)\alpha^L$. As $v^\Psi(\{i,j\}) = (s^h - s^l)\alpha^H = v^\Psi(N) - v^\Psi(N\backslash\{i,j\})$ Furthermore, since

$$v(N) - v(N\backslash N_l^L) = |N_l^L|(s^h - s^l)\alpha^H,$$

it follows that $x_i \leq (s^h - s^l)\alpha^H$.

Lastly, from Definition 6.3.5 we have

$$\theta_i(\Psi) = (s^h - s_i)\alpha_i + \begin{cases} \frac{(\alpha^H - \alpha^L)(s^h - s^l)}{2} & \text{if } i \in N_h^H \cup N_l^L \\ 0 & \text{if } i \in N_l^H \cup N_h^L. \end{cases}$$

Now $\theta(\Psi) = \frac{1}{2}(\overline{\theta}(\Psi) + \underline{\theta}(\Psi))$ follows directly from the definition of $\overline{\theta}(\Psi)$ and $\underline{\theta}(\Psi)$.$\square$

Not only for JiT sequencing situations with $|N_h^H| = |N_l^L| > 1$ we can find an easy expression for the structure of the core of the corresponding game. If the player set contains at least one player of every type and $|N_h^H| \neq |N_l^L|$, the large instance based allocation rule turns out to be the only core element.

Theorem 6.3.9 Let $\Psi = (N, \alpha, s, s_0) \in JiT_h^{2,2}$ be such that N_h^H, N_l^H, N_h^L and N_l^L are all nonempty, and $|N_h^H| \neq |N_l^L|$. Then $\theta(\Psi)$ is the only core element of (N, v^Ψ) and, consequently, $\theta(\Psi) = \eta(v^\Psi)$.

Proof: For every $x \in C(v^\Psi)$ and $i \in N$ it holds that $x_i \leq v^\Psi(N) - v^\Psi(N\backslash\{i\})$. By Theorem 6.3.7, $\theta(\Psi) \in C(v^\Psi)$. It suffices to show that $\theta_i(\Psi) = v^\Psi(N) - v^\Psi(N\backslash\{i\})$ for every $i \in N$, as this implies that for every $x \in C(v^\Psi)$, $x \neq \theta(\Psi)$ we have $\sum_{i\in N} x_i < \sum_{i\in N} \theta(\Psi) = v(N)$ which contradicts the core condition $\sum_{i\in N} x_i = v(N)$.
First consider the case $|N_h^H| > |N_l^L|$. By Corollary 6.3.3 and Definition 6.3.5 we obtain that

$$\begin{aligned} v^\Psi(N) - v^\Psi(N\backslash\{i\}) &= \begin{cases} 0 & \text{if } i \in N_h^H; \\ (s^h - s^l)\alpha^H & \text{if } i \in N_l^L; \\ (s^h - s^l)\alpha^H & \text{if } i \in N_l^H; \\ 0 & \text{if } i \in N_h^L; \end{cases} \\ &= \theta_i(\Psi). \end{aligned}$$

Now consider the case $|N_l^L| > |N_h^H|$. Then, by Corollary 6.3.3 and Definition 6.3.5,

$$\begin{aligned} v^\Psi(N) - v^\Psi(N\backslash\{i\}) &= \begin{cases} (s^h - s^l)(\alpha^H - \alpha^L) & \text{if } i \in N_h^H; \\ (s^h - s^l)\alpha^L & \text{if } i \in N_l^L; \\ (s^h - s^l)\alpha^H & \text{if } i \in N_l^H; \\ 0 & \text{if } i \in N_h^L; \end{cases} \\ &= \theta_i(\Psi). \end{aligned}$$

□

The coincidence between the large instance based allocation rule θ and the nucleolus can be extended to JiT sequencing situations as considered in Theorem 6.3.8 with one further restriction.

Theorem 6.3.10 Let $\Psi = (N, \alpha, s, s_0) \in JiT_h^{2,2}$ be a JiT sequencing situation such that $|N_h^H| = |N_l^L| > 1$. Then $\theta(\Psi) = \eta(v^\Psi)$.

Proof: We show that $\theta(\Psi) = \eta(v^\Psi)$ by showing that $\omega(\theta(\Psi)) \leq_L \omega(x)$ for every $x \in C(v^\Psi)$. By Theorem 6.3.8, $C(v^\Psi) = Conv\{\overline{\theta}(\Psi), \underline{\theta}(\Psi)\}$. So, take $c \in [0,1]$ and define $x^c = c\overline{\theta}(\Psi) + (1-c)\underline{\theta}(\Psi)$. By Lemma 6.3.2, the excesses are

$$E(S,x^c)=\begin{cases}-c(|S_h^H|-|S_l^L|)A & \text{if } S_h^H\neq\emptyset \text{ and } |S_h^H|\geq|S_l^L|;\\ -(1-c)(|S_l^L|-|S_h^H|)A & \text{if } S_h^H\neq\emptyset \text{ and } |S_h^H|<|S_l^L|;\\ -(1-c)|S_l^L|A & \text{if } S_h^H=\emptyset, S_l^L\neq\emptyset \text{ and } S_h^L\neq\emptyset;\\ -(1-c)|S_l^L|A-(s^h-s^l)\alpha^L & \text{if } S_h^H=\emptyset, S_l^L\neq\emptyset \text{ and } S_h^L=\emptyset;\\ -A & \text{if } S_h^H=\emptyset, S_l^L=\emptyset, S_l^H\neq\emptyset \text{ and } S_h^L\neq\emptyset;\\ -(s^h-s^l)\alpha^H & \text{if } S_h^H=\emptyset, S_l^L=\emptyset, S_l^H\neq\emptyset \text{ and } S_h^L=\emptyset;\\ 0 & \text{if } S_h^H=\emptyset, S_l^L=\emptyset \text{ and } S_l^H=\emptyset,\end{cases}$$

where $A=(s^h-s^l)(\alpha^H-\alpha^L)>0$.
It is readily checked that the highest excess equals 0. This excess occurs, independent of the value of c, for every coalition $S\in\mathcal{N}$ such that either $|S_h^H|=|S_l^L|>0$ or $S=S_h^L$. For $c=0$ or $c=1$, there are additional coalitions with excess equal to zero whereas for $c\in(0,1)$ all other coalitions have a negative excess. Hence, both $x^0\neq\eta(v^\Psi)$ and $x^1\neq\eta(v^\Psi)$. Since $-(s^h-s^l)\alpha^H<0$ and $-(s^h-s^l)\alpha^L<0$, for $c\in(0,1)$ the second highest excess equals either $-cA$ or $-(1-c)A$, or a multiple of these values. Hence, the second highest excess is minimized for $c=\frac{1}{2}$, implying that $\eta(v)=x^{\frac{1}{2}}=\theta(\Psi)$. $\square$

In general, the large instance based allocation rule does not coincide with the nucleolus. In fact, the general expression for the nucleolus becomes quite involved as it depends not only on the number of players of every type as in the definition of $\theta(N,\alpha,s,s_0)$, but on the specific values for s^h, s^l, α^H, and α^L as well.

Example 6.3.11 Consider a JiT sequencing situation $\Psi=(N,\alpha,s,s_0)$ such that $|N_h^H|=0$, $|N_l^L|=0$, $|N_l^H|>1$ and $|N_h^L|=1$. By Lemma 6.3.2 we have

$$v^\Psi(N)=(|N_l^H|-1)(s^h-s^l)\alpha^H+(s^h-s^l)\alpha^L.$$

Since all players in N_l^H are symmetric and the nucleolus satisfies symmetry, we have that the nucleolus is of the form

$$x_i^\mu(v^\Psi)=\begin{cases}\frac{1}{|N_l^H|}(v^\Psi(N)-\mu) & \text{if } i\in N_l^H;\\ \mu & \text{if } i\in N_h^L,\end{cases}$$

for some $\mu\in\mathbb{R}$. Let j be the unique element of N_h^L. The excesses are

$$E(S, x^\mu) = \begin{cases} 0 & \text{if } S = \emptyset; \\ \frac{|S_l^H|}{|N_l^H|}\big((s^h - s^l)(\alpha^H - \alpha^L) + \mu\big) - (s^h - s^l)\alpha^H & \text{if } S_l^H \neq \emptyset \text{ and } j \notin S; \\ \frac{|S_l^H| - |N_l^H|}{|N_l^H|}\big((s^h - s^l)(\alpha^H - \alpha^L) + \mu\big) & \text{if } S_l^H \neq \emptyset \text{ and } j \in S; \\ -\mu & \text{if } S_l^H = \emptyset \text{ and } j \in S; \end{cases}$$

Using a similar approach as in the proof of Theorem 6.3.10, we obtain that are three relevant excesses, given by $-\mu$, $-(s^h-s^l)\alpha^L+\mu$ and $-\frac{1}{|N_l^H|}((s^h-s^l)(\alpha^H-\alpha^L)+\mu)$. For $|N| \leq \frac{2\alpha^H}{\alpha^L}$, the maximum of these excesses is minimized by taking $\mu = \frac{1}{2}(s^h - s^l)\alpha^L$. If $|N| > \frac{2\alpha^H}{\alpha^L}$, $\mu = \frac{|N_l^H|}{|N_l^H|+1|}(s^h - s^l)\alpha^H$ minimizes the maximum of these excesses. Hence, we have

$$\eta_i(v^\Psi) = \begin{cases} \frac{1}{2}(s^h - s^l)\alpha^L & \text{if } i = j \text{ and } |N| \leq \frac{2\alpha^H}{\alpha^L}; \\ (s^h - s^l)(\alpha^L - \frac{\alpha^H}{|N_l^H|+1}) & \text{if } i \in N_l^H \text{ and } |N| > \frac{2\alpha^H}{\alpha^L}; \\ (s^h - s^l)(\alpha^H \frac{|N_l^H|-1}{|N_l^H|} + \frac{\alpha^L}{2|N_l^H|}) & \text{if } i \in N_l^H \text{ and } |N| \leq \frac{2\alpha^H}{\alpha^L}; \\ \frac{|N_l^H|}{|N_l^H|+1|}(s^h - s^l)\alpha^H & \text{if } i = j \text{ and } |N| > \frac{2\alpha^H}{\alpha^L}. \end{cases}$$

It is readily checked that $\theta(\Psi) = x^0$, i.e., it distributes $v^\Psi(N)$, irrespective of the specific values of the parameters α^H and α^L, equally over the players in N_l^H, while the unique player in N_h^L obtains 0. $\triangleleft$

6.A Proof of Lemma 6.3.2

Here, we proof the remaining cases of Lemma 6.3.2. For the convenience of the reader, we repeat the lemma and the first part of the proof.

Lemma 6.A.1 *Let* $\Psi = (N, \alpha, s, s_0) \in JiT_h^{2,2}$. *Then*

$$v^{\Psi}(S) = \begin{cases} (|S_l^H| + |S_l^L|)(s^h - s^l)\alpha^H & \text{if } S_h^H \neq \emptyset \text{ and } |S_h^H| \geq |S_l^L|; \\ |S_h^H|(s^h - s^l)(\alpha^H - \alpha^L) + |S_l^H|(s^h - s^l)\alpha^H & \text{if } S_h^H \neq \emptyset \text{ and } |S_h^H| < |S_l^L|; \\ \quad + |S_l^L|(s^h - s^l)\alpha^L & \\ |S_l^H|(s^h - s^l)\alpha^H + |S_l^L|(s^h - s^l)\alpha^L & \text{if } S_h^H = \emptyset, S_l^L \neq \emptyset \\ & \text{and } S_h^L \neq \emptyset; \\ |S_l^H|(s^h - s^l)\alpha^H + (|S_l^L| - 1)(s^h - s^l)\alpha^L & \text{if } S_h^H = \emptyset, S_l^L \neq \emptyset \\ & \text{and } S_h^L = \emptyset; \\ (|S_l^H| - 1)(s^h - s^l)\alpha^H + (s^h - s^l)\alpha^L & \text{if } S_h^H = \emptyset, S_l^L = \emptyset, S_l^H \neq \emptyset \\ & \text{and } S_h^L \neq \emptyset; \\ (|S_l^H| - 1)(s^h - s^l)\alpha^H & \text{if } S_h^H = \emptyset, S_l^L = \emptyset \\ & \text{and } S_h^L = \emptyset; \\ 0 & \text{if } S_h^H = \emptyset, S_l^L = \emptyset \\ & \text{and } S_l^H = \emptyset, \end{cases}$$

for all $S \in \mathcal{N}$.

Proof: First of all, we have

$$\sum_{i \in S} \gamma_i(\sigma^*_{\{i\}}) = (|S_h^H| + |S_l^H|)s^h\alpha^H + (|S_h^L| + |S_l^L|)s^h\alpha^L,$$

for every $S \in \mathcal{N}$.

$S_h^H \neq \emptyset$ *and* $|S_h^H| \geq |S_l^L|$.

Take $S \in \mathcal{N}$ such that $S_h^H \neq \emptyset$ and $|S_h^H| \geq |S_l^L|$. Since $S_h^H \neq \emptyset$, it holds for every (optimal) order $\tilde{\sigma}_S$ provided by Algorithm 1 that $s_{\sigma^*_S(|S|)} = s^h$. Furthermore, since $S_h^H \neq \emptyset$ we have by Proposition 6.2.5 that either $|M^{hH}(\tilde{\sigma}_S)| = 0$ or $|M^{lL}(\tilde{\sigma}_S)| = 0$. It must hold that $|M^{lL}(\tilde{\sigma}_S)| = 0$, since $|S_h^H| \geq |S_l^L|$ together with (6.1) and (6.3) implies that $|M^{hH}(\tilde{\sigma}_S)| \geq |M^{lL}(\tilde{\sigma}_S)|$. So, we have

$$\begin{aligned} \gamma_S(\tilde{\sigma}_S) &= |M^{hH}(\tilde{\sigma}_S)|s^h\alpha^H + |M^{lH}(\tilde{\sigma}_S)|s^l\alpha^H + |M^{hL}(\tilde{\sigma}_S)|s^h\alpha^L + |M^{lL}(\tilde{\sigma}_S)|s^l\alpha^L \\ &= (|S_h^H| - |S_l^L|)s^h\alpha^H + (|S_l^H| + |S_l^L|)s^l\alpha^H + (|S_h^L| + |S_l^L|)s^h\alpha^L, \end{aligned}$$

and we may conclude that

$$\begin{aligned}
v^{\Psi}(S) &= \sum_{i\in S}\gamma_i(\tilde{\sigma}_{\{i\}}) - \gamma_S(\tilde{\sigma}_S) \\
&= (|S_h^H| + |S_l^H|)s^h\alpha^H + (|S_h^L| + |S_l^L|)s^h\alpha^L, \\
&- \big((|S_h^H| - |S_l^L|)s^h\alpha^H + (|S_l^H| + |S_l^L|)s^l\alpha^H + (|S_h^L| + |S_l^L|)s^h\alpha^L\big) \\
&= (|S_l^H| + |S_l^L|)(s^h - s^l)\alpha^H.
\end{aligned}$$

$S_h^H \neq \emptyset$ and $|S_h^H| < |S_l^L|$.
Take $S \in \mathcal{N}$ such that $S_h^H \neq \emptyset$ and $|S_h^H| < |S_l^L|$. Again, for every (optimal) order $\tilde{\sigma}_S$ provided by Algorithm 1 we have $s_{\sigma_S^*(|S|)} = s^h$, and either $|M^{hH}(\tilde{\sigma}_S)| = 0$ or $|M^{lL}(\tilde{\sigma}_S)| = 0$. It must hold that $|M^{hH}(\tilde{\sigma}_S)| = 0$, since $|S_h^H| < |S_l^L|$ together with (6.2) and (6.4) implies that $|M^{lL}(\tilde{\sigma}_S)| > |M^{hH}(\tilde{\sigma}_S)|$. So, we have

$$\begin{aligned}
\gamma_S(\tilde{\sigma}_S) &= |M^{hH}(\tilde{\sigma}_S)|s^h\alpha^H + |M^{lH}(\tilde{\sigma}_S)|s^l\alpha^H + |M^{hL}(\tilde{\sigma}_S)|s^h\alpha^L + |M^{lL}(\tilde{\sigma}_S)|s^l\alpha^L \\
&= (|S_l^H| + |S_h^H|)s^l\alpha^H + (|S_h^H| + |S_h^L|)s^h\alpha^L + (|S_l^L| - |S_h^H|)s^l\alpha^L.
\end{aligned}$$

This gives

$$\begin{aligned}
v^{\Psi}(S) &= \sum_{i\in S}\gamma_i(\tilde{\sigma}_{\{i\}}) - \gamma_S(\tilde{\sigma}_S) \\
&= (|S_h^H| + |S_l^H|)s^h\alpha^H + (|S_h^L| + |S_l^L|)s^h\alpha^L, \\
&- \big((|S_l^H| + |S_h^H|)s^l\alpha^H + (|S_h^H| + |S_h^L|)s^h\alpha^L + (|S_l^L| - |S_h^H|)s^l\alpha^L\big) \\
&= |S_h^H|(s^h - s^l)(\alpha^H - \alpha^L) + |S_l^H|(s^h - s^l)\alpha^H + |S_l^L|(s^h - s^l)\alpha^L.
\end{aligned}$$

$S_h^H = \emptyset$, $S_l^L \neq \emptyset$, and $S_h^L \neq \emptyset$.
Take $S \in \mathcal{N}$ such that $S_h^H = \emptyset$, $S_l^L \neq \emptyset$, and $S_h^L \neq \emptyset$. Since $S_h^L \neq \emptyset$, it holds for every (optimal) order $\tilde{\sigma}_S$ provided by Algorithm 1 that $s_{\sigma_S^*(|S|)} = s^h$. Furthermore since $S_l^L \neq \emptyset$, either $|M^{hH}(\tilde{\sigma}_S)| = 0$ or $|M^{lL}(\tilde{\sigma}_S)| = 0$. It must hold that $|M^{hH}(\tilde{\sigma}_S)| = 0$, since $|S_h^H| < |S_l^L|$ together with (6.2) and (6.4) implies that $|M^{lL}(\tilde{\sigma}_S)| > |M^{hH}(\tilde{\sigma}_S)|$. So, we have

$$\begin{aligned}
\gamma_S(\tilde{\sigma}_S) &= |M^{hH}(\tilde{\sigma}_S)|s^h\alpha^H + |M^{lH}(\tilde{\sigma}_S)|s^l\alpha^H + |M^{hL}(\tilde{\sigma}_S)|s^h\alpha^L + |M^{lL}(\tilde{\sigma}_S)|s^l\alpha^L \\
&= |S_l^H|s^l\alpha^H + |S_h^L|s^h\alpha^L + |S_l^L|s^l\alpha^L.
\end{aligned}$$

We obtain

$$
\begin{aligned}
v^{\Psi}(S) &= \sum_{i \in S} \gamma_i(\tilde{\sigma}_{\{i\}}) - \gamma_S(\tilde{\sigma}_S) \\
&= |S_l^H| s^h \alpha^H + (|S_h^L| + |S_l^L|) s^h \alpha^L, \\
&- (|S_l^H| s^l \alpha^H + |S_h^L| s^h \alpha^L + |S_l^L| s^l \alpha^L) \\
&= |S_l^H| (s^h - s^l) \alpha^H + |S_l^L| (s^h - s^l) \alpha^L.
\end{aligned}
$$

$S_h^H = \emptyset$, $S_l^L \neq \emptyset$ and $S_h^L = \emptyset$.
Take $S \in \mathcal{N}$ such that $S_h^H = \emptyset$, $S_l^L \neq \emptyset$ and $S_h^L = \emptyset$. Since $S_h^H \cup S_h^L = \emptyset$, it holds for every (optimal) order $\tilde{\sigma}_S$ provided by Algorithm 1 that $s_{\sigma_S^*(|S|)} = s^l$. As $S_l^L \neq \emptyset$, Proposition 6.2.5 implies that either $|M^{hH}(\tilde{\sigma}_S)| = 0$ or $|M^{lL}(\tilde{\sigma}_S)| = 0$. It holds that $|M^{hH}(\tilde{\sigma}_S)| = 0$, since $|S_h^H| < |S_l^L|$ together with (6.2) and (6.4) implies that $|M^{lL}(\tilde{\sigma}_S)| \geq |M^{hH}(\tilde{\sigma}_S)|$. So, we have

$$
\begin{aligned}
\gamma_S(\tilde{\sigma}_S) &= |M^{hH}(\tilde{\sigma}_S)| s^h \alpha^H + |M^{lH}(\tilde{\sigma}_S)| s^l \alpha^H + |M^{hL}(\tilde{\sigma}_S)| s^h \alpha^L + |M^{lL}(\tilde{\sigma}_S)| s^l \alpha^L \\
&= |S_l^H| s^l \alpha^H + (|S_h^L| + 1) s^h \alpha^L + (|S_l^L| - 1) s^l \alpha^L,
\end{aligned}
$$

and therefore

$$
\begin{aligned}
v^{\Psi}(S) &= \sum_{i \in S} \gamma_i(\tilde{\sigma}_{\{i\}}) - \gamma_S(\tilde{\sigma}_S) \\
&= |S_l^H| s^h \alpha^H + (|S_h^L| + |S_l^L|) s^h \alpha^L, \\
&- \big(|S_l^H| s^l \alpha^H + (|S_h^L| + 1) s^h \alpha^L + (|S_l^L| - 1) s^l \alpha^L\big) \\
&= |S_l^H| (s^h - s^l) \alpha^H + (|S_l^L| - 1)(s^h - s^l) \alpha^L.
\end{aligned}
$$

$S_h^H = \emptyset$, $S_l^L = \emptyset$, $S_l^H \neq \emptyset$ and $S_h^L \neq \emptyset$.
Take $S \in \mathcal{N}$ such that $S_h^H = \emptyset$, $S_l^L = \emptyset$, $S_l^H \neq \emptyset$ and $S_h^L \neq \emptyset$. Algorithm 1 first places all players in S_h^L but one, then all players in S_l^H and finally the last player in N_h^L. So, we have

$$
\gamma_S(\tilde{\sigma}_S) = (|S_h^L| - 1) s^h \alpha^L + s^h \alpha^H + (|S_l^H| - 1) s^l \alpha^H + s^l \alpha^L,
$$

and we may conclude that

$$
\begin{aligned}
v^{\Psi}(S) &= \sum_{i \in S} \gamma_i(\tilde{\sigma}_{\{i\}}) - \gamma_S(\tilde{\sigma}_S) \\
&= |S_l^H| s^h \alpha^H + |S_h^L| s^h \alpha^L, \\
&- \big((|S_h^L| - 1) s^h \alpha^L + s^h \alpha^H + (|S_l^H| - 1) s^l \alpha^H + s^l \alpha^L\big) \\
&= (|S_l^H| - 1)(s^h - s^l) \alpha^H + (s^h - s^l) \alpha^L.
\end{aligned}
$$

$S_h^H = \emptyset$, $S_l^L = \emptyset$ and $S_h^L = \emptyset$.

Take $S \in \mathcal{N}$ such that $S_h^H = \emptyset$, $S_l^L = \emptyset$ and $S_h^L = \emptyset$. This means $S = S_l^H$ and $\gamma_S(\tilde{\sigma}_S) = (|S_l^H| - 1)s^l\alpha^H + s^h\alpha^H$. Hence,

$$v^\Psi(S) = |S_l^H|s^h\alpha^H - ((|S_l^H| - 1)s^l\alpha^H + s^h\alpha^H) = (|S_l^H| - 1)(s^h - s^l)\alpha^H.$$

$S_h^H = \emptyset$, $S_l^L = \emptyset$ and $S_l^H = \emptyset$.

Take $S \in \mathcal{N}$ such that $S_h^H = \emptyset$, $S_l^L = \emptyset$ and $S_l^H = \emptyset$. This means $S = S_h^L$ and $\gamma_S(\tilde{\sigma}_S) = |S_l^H|s^h\alpha^L$. Hence,

$$v^\Psi(S) = |S_l^H|s^h\alpha^H - |S_l^H|s^h\alpha^H = 0.$$

$\square$

Chapter 7

A taxonomy of rankings in tournaments

7.1 Introduction

In a world full of choices and alternatives, rankings are becoming an increasingly important tool to help individuals and institutions to make decisions. In this chapter we study the classic problem of ranking a series of alternatives when we have information about a number of paired comparisons among them, but possibly not about all paired comparisons. The set of alternatives along with the matrix containing this information will be referred to as a *tournament*. Tournaments appear in a wide variety of real life situations and take different forms. Because of this, the issue of defining rankings over tournaments has been studied in various fields and ranking methods based on different motivations have been defined. Sport events and, in particular, chess, motivated the seminal work on rankings by Zermelo (1929). Later on, sparked by Arrow's impossibility theorem (see Arrow (1953)), this topic emerged in the context of social choice and voting theory. The theory of rankings has also attracted statisticians and psychologists, which have studied it under the name of paired comparisons analysis.

We consider *tournaments* represented by a set of alternatives N and a non-negative $N \times N$ matrix A. Although different interpretations can be given to the entries of the matrix, we stick to the following interpretation throughout this chapter. The entry A_{ij} represents the aggregate result of alternative $i \in N$ in the direct pairwise comparisons with alternative $j \in N$; it is assumed that $A_{ii} = 0$ for every alternative $i \in N$. The results of alternatives i and j in an elementary comparison between these alternatives are non-negative and sum to one. This may indicate a win, loss or draw, but could also be a proxy for the relative strength in the comparison based on, e.g., goals scored in soccer or IMPs in bridge. The sum of the aggregate results, $A_{ij} + A_{ji}$, represents the amount of elementary comparisons between i and j. We allow for multiple elementary comparisons, and also allow $A_{ij} + A_{ji}$ to be non-integer, this way allowing for partially finished matches or, in testing objects, a difference in importance of the comparison of the same alternatives by different experts. We allow for partial information about all possible pairwise comparisons, the amount of comparisons between two alternatives can be zero, since there are many situations where it is unfeasible to obtain complete information. This may be because there are a high number of alternatives to be ranked or just because it is too costly to undergo each pairwise comparison.

In voting theory, tournaments are typically defined through complete and asymmetric binary relations. In our terminology this would correspond with a tournament such that for each pair of alternatives $i, j \in N$, $A_{ij} \in \{0, 1\}$ and $A_{ij} + A_{ji} = 1$. We call such a tournament *binary*. For binary tournaments the score ranking method, which for every alternative equals the sum of its aggregate results against all other alternatives divided by his total number of comparisons, will not tell the whole story due to the partial information about all pairwise comparisons. It seems reasonable to take into account the quality of the alternatives it has been compared with. In the literature on social choice and voting, several overviews of ranking methods and their properties exist, mostly dealing with binary tournaments. Laslier (1997) presents a thorough analysis of different ranking methods and properties defined for binary tournaments.[1] This approach allows for set-valued rankings and the spirit of many of the properties revolve around the (possible) set-valuedness of the ranking methods. Maybe more importantly, the analysis there is focused on the (important) application to voting situations. Bouyssou (2004) revisits the main ranking methods in Laslier (1997) and studies their monotonicity properties.

Our goal in this chapter is to take some of the ranking methods considered in the different fields and compare them by looking at their performance with respect to a set of natural properties. These properties originate from different contexts, and therefore the desirability of these properties can depend on the application. Although our results are context free, we use motivation, terminology and interpretation from sports competitions throughout this chapter for expositional purposes.

[1]Dutta and Laslier (1999) consider a setting where this last restriction is generalized to allow for intensities but completeness is still a requirement.

We conduct a detailed analysis of the properties of several well known ranking methods and the newly introduced recursive Buchholz. Roughly speaking, the recursive Buchholz ranking method adds depth to the ideas underlying Buchholz, as it not only takes the strength of your opponents into account, but also the strength of your opponents' opponents etc.

Our analysis mainly concentrates on five ranking methods. First, the scores ranking method which is a natural choice for binary tournaments (see Rubinstein (1980) for an axiomatic characterization). Second, the fair bets ranking method, which is a ranking method widely studied in social choice and economics (see, for instance, Daniels (1969), Moon and Pullman (1970), Slutzki and Volij (2005) and Slutzki and Volij (2006)). Third, the maximum likelihood ranking method, the most common choice in statistics and psychology (see, for instance, Zermelo (1929) and Bradley and Terry (1952)). Lastly, we study two ranking methods stemming from a new approach developed in Brozos-Vázquez et al. (2008): the recursive performance ranking method as introduced by Brozos-Vázquez et al. (2008) and the newly introduced ranking method recursive Buchholz. Also, as a benchmark for fair bets and recursive Buchholz, we consider the Neustadtl ranking method and the Buchholz ranking method.

The main contribution of this chapter is to study how the above ranking methods perform with respect to a set of natural properties, in a general framework of tournaments. This analysis is important not only to get a better understanding of the different ranking methods, but also to learn about the strength and implications of the different properties. To give two examples, first the property of independence of irrelevant matches clarifies an important difference between the score ranking method and the other ranking methods under consideration, only the score ranking method does not consider the results of one's opponent to rank the players. Second, maximum likelihood behaves well with respect to several properties. This is somewhat surprising since, because of its nature, one would expect maximum likelihood to have good statistical properties (for instance, in terms of asymptotic behavior), but there is no reason to expect good behavior with respect to some of the properties we work with. As a byproduct we can extend the characterization by Slutzki and Volij (2005) of the fair bets ranking method to our domain. This chapter is structured as follows. In Section 7.2 we present the main definitions and ranking methods. In Sections 7.3 to 7.6 we define and discuss several properties. Finally, we discuss the results of our analysis in Section 7.7.

7.2 Tournaments and ranking methods

The framework of tournaments is used to obtain a ranking of a finite set of alternatives based on partial information about direct pairwise comparisons between these alternatives. We first introduce tournaments and a number of related notions. Then, we define and discuss the ranking methods under consideration in this chapter.

We define a *tournament* as a pair (N, A), where N is a finite set of alternatives, or players, and $A \in \mathbb{R}^{N \times N}$ is the tournament matrix. The results of players i and j in an elementary pairwise comparison, or match, between these players sum to one. The tournament matrix A is such that A_{ij} represents the aggregate result of player i over all matches against j. We assume $A_{ij} \geq 0$ for all $i, j \in N$ and $A_{ii} = 0$ for all $i \in N$. To ensure that all ranking methods presented in this paper are well-defined, we make the standard assumption that the matrix A is *irreducible*. This means that for every pair of players $i, j \in N$, $i \neq j$, there has to be a sequence of players $(i = k_0, k_1, \ldots, k_n = j)$ such that, for each $\ell \in \{0, \ldots, n-1\}$, $A_{k_\ell k_{\ell+1}} > 0$.

To each tournament (N, A) we associate a (symmetric) *matches matrix* $M(A) = A + A^\top$. As the results of an elementary comparison sum to one, $M_{ij}(A)$ represents the amount of matches between $i \in N$ and $j \in N$. A tournament is called *round-robin* if $M_{ij} = 1$ for all $i, j \in N, i \neq j$. For each player $i \in N$, define $m_i(A) = \sum_{j \in N} M_{ij}(A)$. So, $m_i(A)$ equals the total amount of matches played by player $i \in N$. For $i, j \in N$, define $\bar{M}_{ij}(A) = \frac{M_{ij}(A)}{m_i(A)}$ to be the proportion of player i's matches that he plays against j.

A *rating method* r assigns to any tournament (N, A) a rating vector $r(A) \in \mathbb{R}^N$ where $r_i(A)$, $i \in N$, is a measure of the performance of player $i \in N$. A *ranking method* φ assigns to any tournament (N, A) a weak order $\varphi(A)$ on N (transitive, complete).

For $i, j \in N$, we write $i \; R^{\varphi(A)} \; j$ if i is ranked weakly above j according to $\varphi(A)$; a strict ranking is denoted by $i \; P^{\varphi(A)} \; j$ and indifference is denoted by $i \; I^{\varphi(A)} \; j$. A weak order $\varphi(A)$ is called *flat* if for each pair $i, j \in N$, $i \; I^{\varphi(A)} \; j$. For a rating method r, the ranking method φ^r assigns to each tournament A the weak order induced by $r(A)$: $i \; R^{\varphi^r(A)} \; j$ if and only if $r_i(A) \geq r_j(A)$.

When no confusion can arise, we use the shorthand notation M for $M(A)$, m for $m(A)$, $\bar{M}$ for $\bar{M}(A)$ and r for $r(A)$.

Below we introduce the corresponding rating methods used to define the ranking methods considered in this chapter. Two of them, the Neustadtl and Buchholz ranking methods are mainly defined as a benchmark for the fair bets and recursive Buchholz methods, respectively.

Definition 7.2.1 The *scores* rating method r^{s} assigns to any tournament (N, A) the rating vector $r^{\mathrm{s}}(A)$ defined by $r^{\mathrm{s}}_i(A) = \sum_{j \in N} A_{ij}/m_i$ for all $i \in N$.

It follows from the assumption that a tournament is irreducible that $r^{\mathrm{s}}_i \in (0, 1)$ for all $i \in N$. We illustrate the ranking methods with the following example.

Example 7.2.2 Consider the tournament A described below, which is an adaptation of an example in Borm et al. (2002) to a non round-robin setting.

$$\begin{array}{ccc} A & M & m \\ \begin{pmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 4 \\ 1 & 1 & 4 & 0 \end{pmatrix} & \begin{pmatrix} 4 \\ 4 \\ 6 \\ 6 \end{pmatrix} \end{array}$$

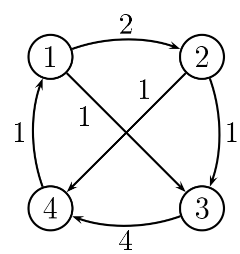

Figure 7.2.1: Graph representation of tournament A of Example 7.2.2

The scores rating vector is given by $r^{\mathrm{s}}(A) = (\frac{3}{4}, \frac{1}{2}, \frac{2}{3}, \frac{1}{6})$, so the scores ranking method $\varphi^{r^{\mathrm{s}}}$ ranks players 1 on top, followed by player 3, player 2 and lastly player 4. $\triangleleft$

Definition 7.2.3 For any tournament (N, A), let the $N \times N$ matrix $\hat{A}$ be defined by $\hat{A}_{ij} = A_{ij}/m_i$ for every pair $i, j \in N$. The *Neustadtl* rating method r^{n} assigns to any tournament (N, A) the rating vector $r^{\mathrm{n}}(A) = \hat{A} r^{\mathrm{s}}(A)$.

For each player $i \in N$, the Neustadtl rating vector is a weighted sum of the scores of all other players, where the weight of player j's score is proportional to the result of player i against player j.[2] Thus, the idea behind Neustadtl is to reward a win against a player with a high score more than a win against a player with a lower score.

For an arbitrary vector $a \in \mathbb{R}^N$, $\text{diag}(a)$ denotes the $N \times N$ matrix given by $\text{diag}_{ii}(a) = a_i$ for every $i \in N$, and $\text{diag}_{ij}(a) = 0$ for every $i, j \in N$, $i \neq j$. For every tournament (N, A), let $L_A = \text{diag}(A^\top e^N)$. So, for every $i \in N$, $(L_A)_{ii}$ represents the aggregate result of all other players against player i.

Definition 7.2.4 The *fair bets* rating method r^{fb} assigns to any tournament (N, A) the rating vector $r^{\text{fb}}(A)$, which is defined as the unique solution to the system of linear equations given by $L_A^{-1} A x = x$ and $x^\top e^N = 1$ or, equivalently, by $\sum_{j \in N} A_{ij} x_j = \sum_{j \in N} A_{ji} x_i$ for all $i \in N$, and $x^\top e^N = 1$.

This method was originally defined for round-robin tournaments and has been studied in a variety of fields under different names and interpretations: from the classic chapters by Daniels (1969) and Moon and Pullman (1970), to more recent references such as Slutzki and Volij (2005) and Slutzki and Volij (2006). Note that for every tournament A, $r^{\text{fb}}(A)$ is positive.

The fair bets ratings are such that, if a bettor wins $A_{ij} r^{\text{fb}}_j(A)$ from the results of player i against player j and loses $A_{ji} r^{\text{fb}}_i(A)$ from the results of player j against player i then for the results of player i total revenue $(\sum_{j \in N} A_{ij} r^{\text{fb}}_j(A))$ equals total loss $(\sum_{j \in N} A_{ji} r^{\text{fb}}_i(A))$.

Ranking methods using similar ideas as the fair bets ranking method are provided by Daniels (1969), Borm et al. (2002), Herings et al. (2005) and Slikker et al. (2011). Daniels (1969) introduces the invariant ranking method, where the underlying rating vector is such that the rating of player i equals $\sum_{j \in N} A_{ij} r^{\text{fb}}_j(A)$. Borm et al. (2002) introduce the lambda ranking method which, as Slikker et al. (2011) show, for tournaments (N, A) such that $M_{ij}(A) \in \{0, 1\}$ for every $i, j \in N$, can be seen as a compromise between the fair bets ranking method and the invariant ranking method. Although Borm et al. (2002) extend the lambda ranking method beyond round-robin tournaments, this is not directly applicable to our domain since they assume that $A_{ii} > 0$ for every $i \in N$.

[2] This rule is commonly known as Sonneborn-Berger, but it was originally proposed by Hermann Neustadtl. Actually, this ranking method was defined just for round-robin tournaments and what we present here is a natural extension to our more general setting. The Neustadtl ranking is often used for breaking ties after applying the scores ranking method.

Both the Neustadtl ranking method and the fair bets ranking method reward good results against players with good results. However, with fair bets, it is not only important to have beaten players who have good results, but also that they have achieved these results against players with good results, and so on. Hence, the fair bets ranking method can be seen as a ranking method adding depth to the Neustadtl ranking method.

Example 7.2.2 (continued) We have

$$\hat{A} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{2}{3} \\ \frac{1}{6} & 0 & 0 & 0 \end{pmatrix},$$

therefore, the Neustadtl rating vector is given by $r^{\mathrm{n}}(A) = (0.417, 0.208, 0.111, 0.125)$, which puts player 1 ranked on top, followed by player 2, player 4 and player 3. Although player 3 has a relatively high score, this was achieved against players with a low score. This makes that he is, contrary to the scores ranking, ranked below player 2 and 4. The fair bets rating vector is given by $r^{\mathrm{fb}}(A) = (0.526, 0.158, 0.211, 0.105)$. So, the fair bets ranking coincides with the scores ranking. In comparison with the Neustadtl ranking, player 3 is ranked higher as he has a good result against player 4, who in turn has a good result against a player with a high score, player 1. $\triangleleft$

Next, we introduce the maximum likelihood ranking method.

Definition 7.2.5 The *maximum likelihood* rating method r^{ml} assigns to any tournament (N, A) the rating vector $r^{\mathrm{ml}}(A)$, defined for every player $i \in N$, by $r_i^{\mathrm{ml}}(A) = \log(\pi_i(A))$. Here, $\pi(A) \in \mathbb{R}^N$ is the unique (positive) solution of the system of non-linear equations given by $\pi(A)^\top e^N = 1$ and, for each $i \in N$,[3]

$$\pi_i(A) = \frac{m_i r_i^{\mathrm{s}}(A)}{\sum_{j \in N \setminus \{i\}} \frac{M_{ij}}{(\pi_i(A) + \pi_j(A))}}.$$

The origins of this ranking method can be traced as far back as Zermelo (1929) and it has also been studied in a wide variety of fields (see, for instance, Bradley

[3] Refer, for instance, to Ford (1957) or David (1988).

and Terry (1952), Moon and Pullman (1970) and David (1988)). The maximum likelihood ranking method originates from the setting where the result of a match is binary. It is assumed that each player $i \in N$ has a rating r_i and that, given two players $i, j \in N$, $i \neq j$, the probability that player i beats player j is given by a *probability function* $F(r_i, r_j)$. In the classic approach, used already in the early works by Zermelo (1929) and rediscovered by Bradley and Terry (1952), F is based on the (standard) logistic distribution $F_L(x) = 1/(1 + \exp(-x))$, so that $F(r_i, r_j) = F_L(r_i - r_j) = \exp(r_i)/(\exp(r_i) + \exp(r_j))$. Then, the maximum likelihood rating method looks for the ratings that maximize the probability of the matrix A being realized when all the matches given in matrix M take place. In our setting, the probability function can be interpreted as the expected result of a match between players $i \in N$ and $j \in N$.

The recursive performance ranking method, defined in Brozos-Vázquez et al. (2008), also builds upon probability functions. Given a tournament (N, A), one would like to associate with it a rating of the players that explains all the observed results, that is, for each pair of players $i, j \in N$, $i \neq j$, $F(r_i, r_j) = \frac{A_{ij}}{M_{ij}}$, *i.e.*, the observed result of i against j is exactly what one would predict using F and the ratings r_i and r_j. Unfortunately, finding such ratings amounts to solving a system with far more equations than variables which, for most tournaments, will have no solution. As we said above, what maximum likelihood does is to find the ratings under which the probability of the observed results is maximized, whereas the recursive performance finds the rating that explains the 'average' result of each player.

Given a tournament (N, A), a rating vector $r \in \mathbb{R}^N$, and a player i, the average opponent of i in the tournament is $(\bar{M}r)_i$, *i.e.*, the average rating of the opponents of i (weighted by the amount of matches played against each of them). The recursive performance looks for a rating such that for each player $i \in N$, $F(r_i, (\bar{M}r)_i) = \sum_{j \in N} A_{ij}/m_i = r^{\mathrm{s}}_i(A)$. Again, we stick to the approach of using the logistic distribution to define F.

Definition 7.2.6 For any tournament (N, A) and every $i \in N$ let $c(A) \in \mathbb{R}^N$ be defined by $c(A)_i = F_L^{-1}(r^{\mathrm{s}}_i(A))$ and let $\hat{c}(A) = c(A) - \frac{m^\top c(A)}{m^\top e^N} e^N$. The *recursive performance* rating method r^{rp} assigns to any tournament (N, A) the rating vector $r^{\mathrm{rp}}(A)$, which is the solution of the system of linear equations given by $x^\top e = 0$ and

$\bar{M}x + \hat{c}(A) = x.$

Hence, for each player i, r_i^{rp} takes into account the average strength of i's opponents ($\bar{M}r^{\text{rp}}$) and his own score in the tournament ($\hat{c}_i$ is increasing in r_i^{s}). Although we define the recursive performance rating method as the solution of a system of equations, the original definition introduces the rating method as the limit of the iterative method $p(0) = \bar{M}r + \hat{c}(A)$, $p(t) = \bar{M}p(t-1) + \hat{c}(A)$ for $t \in \mathbb{N}$, where r is an exogenously given vector of initial ratings. In fact, the limit of this iterative method is independent of the initial rating vector r.

For round-robin tournaments, Buchholz is a tie-breaking rule that ranks players with an equal score according to the average scores of their opponents. To keep this idea we introduce the Buchholz ranking method, that combines the average strength of one's opponent and one's own score. Thus, Buchholz uses the same ideas as recursive performance, but in a much simpler way.

Definition 7.2.7 The *Buchholz* rating method r^{b} assigns to any tournament (N, A) the rating vector $r^{\text{b}}(A)$, given by $r^{\text{b}}(A) = \bar{M}r^{\text{s}}(A) + r^{\text{s}}(A)$.

Recursive Buchholz is a new ranking method that combines the ideas of Buchholz and recursive performance by adding to the Buchholz ranking method the same kind of depth that the fair bets added to Neustadtl. Not only the average score of your opponents ($\bar{M}r^{\text{s}}(A)$) should be important, but also whether your opponents have achieved this average score against weak or strong opponents. All else equal, having faced opponents with a high score who have themselves played against strong opponents should be better than having faced opponents with a high score who have played against weak opponents. Again, further depth can be given to this argument and recursive Buchholz captures this idea.

Definition 7.2.8 The *recursive Buchholz* rating method r^{rb} assigns to any tournament (N, A) the rating vector $r^{\text{rb}}(A)$, defined as the unique solution of the system of linear equations given by $x^\top e^N = 0$ and $\bar{M}x + \hat{r^{\text{s}}}(A) = x$, where $\hat{r^{\text{s}}}(A) = r^{\text{s}}(A) - \frac{e^N}{2}$.[4]

[4] Since the recursive Buchholz can be seen as a variation of the recursive performance where F_L is taken to be the identity, the existence and uniqueness of r^{rb} follows from Theorem 2 in Brozos-Vázquez et al. (2008)

For the recursive Buchholz ranking method the same remark holds as for the maximum likelihood ranking method: although it is defined as the solution to a system of equations, it can also be obtained as the limit of the iterative method $p(0) = \bar{M}r + \hat{r^{\mathrm{s}}}(A)$, $p(t) = \bar{M}p(t-1) + \hat{r^{\mathrm{s}}}(A)$ for $t \in \mathbb{N}$, where r is an exogenously given rating vector. In fact, the limit of this iterative method is independent of the rating vector r.

Example 7.2.2 (continued) We have $\pi(A) = (0.512, 0.236, 0.205, 0.047)$, so the maximum likelihood rating vector is given by $r^{\mathrm{ml}}(A) = (-0.669, -1.444, -1.587, -3.055)$. Hence, player 1 is ranked on top, followed by player 2, player 3 and player 4. For the remaining rating methods, we have $r^{\mathrm{rp}}(A) = (0.971, 0.238, 0.086, -1.295)$, $r^{\mathrm{b}}(A) = (1.208, 1.083, 0.986, 0.819)$ and $r^{\mathrm{rb}}(A) = (0.483, 0.317, 0.300, 0)$. Hence, $\varphi^{r^{\mathrm{ml}}}(A)$, $\varphi^{r^{\mathrm{rp}}}(A)$, $\varphi^{r^{\mathrm{b}}}(A)$ and $\varphi^{r^{\mathrm{rb}}}(A)$ coincide. $\triangleleft$

In the remainder, we use the shorthand notation φ^{s} for $\varphi^{r^{\mathrm{s}}}$. For the other ranking methods, we use an equivalent shorthand notation.

In the following sections we discuss several properties and study whether the above ranking methods satisfy them. Most of the properties we discuss have been studied in the literature before. We will be explicit when defining properties that we have not found in the literature. We only consider ordinal properties, relating to the ranking of the players and not necessarily to the underlying rating vector.

7.3 Basic properties

In this section we start our analysis by presenting three elementary properties that a ranking method φ can satisfy. In addition, we present a property that deals with bridge players and the sub-tournaments in which such players naturally divide a given tournament.

Definition 7.3.1 The ranking method φ satisfies *anonymity* (ANO) if for any tournament (N, A), any $i, j \in N$ and (N, A') the tournament obtained from (N, A) by permuting columns i and j and rows i and j, $\varphi(A)$ and $\varphi(A')$ are the same but with players i and j interchanged.

Definition 7.3.2 The ranking method φ satisfies *homogeneity* (HOM) if $\varphi(kA) = \varphi(A)$ for all tournaments (N, A) and all $k > 0$.

Definition 7.3.3 A tournament (N, A) is symmetric if $A = A^\top$. The ranking method φ satisfies *symmetry* (SYM) if $\varphi(A)$ is flat for any symmetric tournament (N, A).

These three properties require no motivation. It is readily verified that all our ranking methods satisfy these three properties.

Theorem 7.3.4 The ranking methods φ^{s}, φ^{n}, φ^{fb}, φ^{ml}, φ^{rp}, φ^{b}, and φ^{rb} all satisfy ANO, HOM and SYM.

Given a tournament (N, A), a player $b \in N$ is a *bridge player* if there exist $N^1, N^2 \subseteq N$ with $|N^1| \geq 2$ and $|N^2| \geq 2$ such that $N^1 \cup N^2 = N$, $N^1 \cap N^2 = \{b\}$ and $M_{ij} = 0$ for all $i \in N^1 \backslash \{b\}, j \in N^2 \backslash \{b\}$. The sub-tournaments (N^1, A^1) and (N^2, A^2) are such that $A^1_{ij} = A_{ij}$ for every $i, j \in N^1$, and $A^2_{ij} = A_{ij}$ for every $i, j \in N^2$. Since no player of $N^1 \backslash \{b\}$ has played against any player in $N^2 \backslash \{b\}$, the irreducibility of a tournament with bridge players depends crucially on these bridge players. The sub-tournaments (N^1, A^1) and (N^2, A^2) satisfy our definition of a tournament. By irreducibility of (N, A) and the observation that $A_{ji} = A_{ij} = 0$ for every $i \in N^1 \backslash \{b\}$ and $j \in N \backslash N^1$, it follows that for every pair $i, j \in N^1$, $i \neq j$ we can find a sequence of players $(i = k_0, k_1, ..., k_n = j)$ such that $k_l \in N^1$ for each $l \in \{1, ..., n-1\}$ and, for each $l \in \{0, ..., n-1\}$, $A_{k_l k_{l+1}} > 0$. As $A_{ij} = A^1_{ij}$ for every $i, j \in N^1$ this means that (N^1, A^1) is irreducible. Note that, given a tournament (N, A) and a bridge player $b \in N$, the associated sub-tournaments (N^1, A^1) and (N^2, A^2) need not be unique.

Definition 7.3.5 The ranking method φ satisfies *sub-tournament separability* (SS) if for each tournament (N, A), every bridge player $b \in N$, and any associated sub-tournaments (N^1, A^1) and (N^2, A^2), $i \ R^{\varphi(A)} \ j$ if and only if $i \ R^{\varphi(A^1)} \ j$ for all $i, j \in N^1$.

With the presence of a bridge player a tournament is irreducible, but every comparison across the sub-tournaments has to be done via the bridge player. So, the relative strength of the players in one tournament with respect to the players in the other tournament can only be done by using their performance against the bridge player as a reference point. Assuming that the bridge player plays with the same strength in both tournaments, sub-tournament separability states that the results in one part of the tournament does not affect the relative ranking of the players in the other part of the tournament.

As the following theorems show, three ranking methods satisfy SS: fair bets, maximum likelihood and recursive Buchholz.

Theorem 7.3.6 φ^{fb} satisfies SS.

Proof: Let (N, A) be a tournament and let $b \in N$ be a bridge player with respect to the subtournaments (N^1, A^1) and (N^2, A^2). Take $x^1 = r^{\text{fb}}(N^1, A^1)$ and $x^2 = r^{\text{fb}}(N^2, A^2)$ and define, for all $i \in N$,

$$y_i = \begin{cases} a\frac{x_b^2}{x_b^1}x_i^1 & \text{if } i \in N^1 \\ ax_i^2 & \text{if } i \in N^2 \end{cases},$$

where

$$a = \frac{1}{(\sum_{i\in N^1} \frac{x_b^2}{x_b^1}x_i^1 + \sum_{i \in N^2\backslash\{b\}} x_i^2)}.$$

Since the fair bets rating vector associated with an irreducible tournament is positive, the vector y is well defined. Clearly, $\sum_{i\in N} y_i = 1$. Also, for $i \in N^1\backslash\{b\}$ we have that, for all $j \in N^2$, $A_{ij} = A_{ji} = 0$ and hence

$$\sum_{j\in N} A_{ij}y_j = \sum_{j\in N^1} A_{ij}^1 a\frac{x_b^2}{x_b^1}x_j^1 = a\frac{x_b^2}{x_b^1}\sum_{j\in N^1} A_{ji}^1 x_j^1 = a\frac{x_b^2}{x_b^1}\sum_{j\in N^1} A_{ji}^1 x_i^1 = \sum_{j\in N} A_{ji}y_i.$$

Similarly, for $i \in N^2\backslash\{b\}$ we have

$$\sum_{j\in N} A_{ij}y_j = \sum_{j\in N^2} A_{ij}^2 a x_j^2 = \sum_{j\in N^2} A_{ji}^2 a x_i^2 = \sum_{j\in N} A_{ji}y_i.$$

Finally,

$$\begin{aligned}
\sum_{j\in N} A_{bj}y_j &= \sum_{j\in N^1} A^1_{bj} a\frac{x^2_b}{x^1_b}x^1_j + \sum_{j\in N^2} A^2_{bj} a x^2_j \\
&= a\frac{x^2_b}{x^1_b}\sum_{j\in N^1} A^1_{bj}x^1_j + a\sum_{j\in N^2} A^2_{bj}x^2_j \\
&= a\frac{x^2_b}{x^1_b}\sum_{j\in N^1} A^1_{jb}x^1_b + a\sum_{j\in N^2} A^2_{jb}x^2_b \\
&= \sum_{j\in N} A_{jb}y_b.
\end{aligned}$$

Hence, $y = r^{\mathrm{fb}}(N,A)$. Also, y and x^1 resp. x^2 induce the same rankings on the players in N^1 resp. N^2. From this, SS follows. □

Theorem 7.3.7 φ^{ml} satisfies SS.

Proof: Let (N,A) be a tournament and let $b \in N$ be a bridge player with respect to the subtournaments (N^1,A^1) and (N^2,A^2). Take for all $i \in N$, $x^1_i = \pi_i(N^1,A^1) = \exp(r^{\mathrm{ml}}_i(N^1,A^1))$ and $x^2_i = \pi_i(N^2,A^2) = \exp(r^{\mathrm{ml}}_i(N^2,A^2))$ and define

$$y_i = \begin{cases} a\frac{x^2_b}{x^1_b}x^1_i & \text{if } i \in N^1 \\ a x^2_i & \text{if } i \in N^2 \end{cases},$$

where

$$a = \frac{1}{(\sum_{i\in N^1}\frac{x^2_b}{x^1_b}x^1_i + \sum_{i\in N^2\setminus\{b\}} x^2_i)}.$$

Since the maximum likelihood rating vector associated with an irreducible tournament is positive, the vector y is well defined. Clearly, $\sum_{i\in N} y_i = 1$. Then, for $i \in N^1\setminus\{b\}$ we have that, for all $j \in N^2$, $A_{ij} = A_{ji} = 0$ and hence

$$\begin{aligned}
y_i &= a\frac{x^2_b}{x^1_b}x^1_i \\
&= a\frac{x^2_b}{x^1_b}\frac{m^1_i s^1_i}{\sum_{j\in N^1\setminus\{i\}}\frac{M^1_{ij}}{x^1_i+x^1_j}} \\
&= \frac{m_i s_i}{\sum_{j\in N\setminus\{i\}}\frac{M_{ij}}{a\frac{x^2_b}{x^1_b}x^1_i + a\frac{x^2_b}{x^1_b}x^1_j}} \\
&= \frac{m_i s_i}{\sum_{j\in N\setminus\{i\}} y_i + y_j}.
\end{aligned}$$

For $i \in N^2 \backslash \{b\}$ it is trivial that $y_i = \frac{m_i s_i}{\sum_{j \in N \backslash \{i\}} y_i + y_j}$. Finally,

$$\begin{aligned}
y_b &= a x_b^2 \\
&= \frac{m_b^1 s_b^1 + m_b^2 s_b^2}{\frac{m_b^1 s_b^1}{a \frac{x_b^2}{x_b^1} x_b^1} + \frac{m_b^2 s_b^2}{a x_b^2}} \\
&= \frac{m_b^1 s_b^1 + m_b^2 s_b^2}{\sum_{j \in N^1 \backslash \{b\}} \frac{M_{bj}^1}{a \frac{x_b^2}{x_b^1}(x_b^1 + x_j^1)} + \sum_{j \in N^2 \backslash \{b\}} \frac{M_{bj}^2}{a x_b^2 + a x_j^2}} \\
&= \frac{m_b s_b}{\sum_{j \in N^1 \backslash \{b\}} \frac{M_{bj}^1}{y_b + y_j} + \sum_{j \in N^2 \backslash \{b\}} \frac{M_{bj}^2}{y_b + y_j}} \\
&= \frac{m_b s_b}{\sum_{j \in N \backslash \{b\}} \frac{M_{bj}}{y_b + y_j}}.
\end{aligned}$$

Hence, $y = r^{\mathrm{ml}}(N, A)$. Also, y and x^1 resp. x^2 induce the same rankings on the players in N^1 resp. N^2. From this, SS follows. □

Theorem 7.3.8 φ^{rb} satisfies SS.

Proof: Let (N, A) be a tournament and let $b \in N$ be a bridge player with respect to the subtournaments (N^1, A^1) and (N^2, A^2). Take for every $i \in N$, $x_i^1 = r_i^{\mathrm{rb}}(N^1, A^1)$ and $x_i^2 = r_i^{\mathrm{rb}}(N^2, A^2)$ and define

$$y_i = \begin{cases} x_i^1 + x_b^2 - a & \text{if } i \in N^1 \\ x_b^1 + x_i^2 - a & \text{if } i \in N^2, \end{cases}$$

where $a = \frac{1}{|N|}(\sum_{i \in N^1}(x_i^1 + x_b^2) + \sum_{i \in N^2 \backslash \{b\}}(x_b^1 + x_i^2))$. Clearly, $\sum_{i \in N} y_i = 0$. Also, for $i \in N^1 \backslash \{b\}$ we have that, for all $j \in N^2$, $A_{ij} = A_{ji} = 0$ so $M_{ij} = 0$ and hence

$$
\begin{aligned}
y_i &= x_i^1 + x_b^2 - a \\
&= \left(\sum_{j \in N^1} \frac{M_{ij}^1}{m_i^1} x_j^1 \right) + \hat{r^{\mathrm{s}}}_i^1 + x_b^2 - a \\
&= \sum_{j \in N^1} \frac{M_{ij}^1}{m_i^1} (x_j^1 + x_b^2 - a) + \hat{r^{\mathrm{s}}}_i^1 \\
&= \sum_{j \in N} \frac{M_{ij}}{m_i} (x_j^1 + x_b^2 - a) + \hat{r^{\mathrm{s}}}_i \\
&= \sum_{j \in N} \frac{M_{ij}}{m_i} y_j + \hat{r^{\mathrm{s}}}_i,
\end{aligned}
$$

where the third equality follows from $\sum_{j \in N^1} \frac{M_{ij}^1}{m_i^1} = 1$. For $i \in N^2 \backslash \{b\}$ we obtain in the same way that $y_i = \sum_{j \in N} \frac{M_{ij}}{m_i} y_j + \hat{r^{\mathrm{s}}}_i$. Finally,

$$
\begin{aligned}
y_b &= x_b^1 + x_b^2 - a \\
&= \frac{m_b^1}{m_b}(x_b^1 + x_b^2 - a) + \frac{m_b^2}{m_b}(x_b^1 + x_b^2 - a) \\
&= \frac{m_b^1}{m_b} \left(\sum_{j \in N^1} \frac{M_{bj}^1}{m_b^1} x_j^1 + \hat{r^{\mathrm{s}}}_b^1 + x_b^2 - a \right) \\
&\quad + \frac{m_b^2}{m_b} \left(\sum_{j \in N^2} \frac{M_{bj}^2}{m_b^2} x_j^2 + \hat{r^{\mathrm{s}}}_b^2 + x_b^1 - a \right) \\
&= \frac{m_b^1}{m_b} \left(\sum_{j \in N^1} \frac{M_{bj}^1}{m_b^1} (x_j^1 + x_b^2 - a) + \hat{r^{\mathrm{s}}}_b^1 \right) \\
&\quad + \frac{m_b^2}{m_b} \left(\sum_{j \in N^2} \frac{M_{bj}^2}{m_b^2} (x_j^2 + x_b^1 - a) + \hat{r^{\mathrm{s}}}_b^2 \right) \\
&= \sum_{j \in N} \frac{M_{bj}}{m_b} y_j + \hat{r^{\mathrm{s}}}_b,
\end{aligned}
$$

where the last equality follows from $\hat{r^{\mathrm{s}}}_b = \frac{m_b^1}{m_b} \hat{r^{\mathrm{s}}}_b^1 + \frac{m_b^2}{m_b} \hat{r^{\mathrm{s}}}_b^2$. So, we have $y = r^{\mathrm{rb}}(N, A)$. y and x^1 resp. x^2 induce the same rankings on the players in N^1 resp. N^2. From this, SS follows. □

The other ranking methods under consideration however do not satisfy SS. This is shown in the following example.

Example 7.3.9 Consider the tournaments (N, A) and (N^1, A^1) where $N = \{1, 2, 3, 4\}$, $N^1 = \{1, 2, 3\}$ and A, A^1 are as described below:

$$\begin{array}{c} A \\ \begin{pmatrix} 0 & 1 & 20 & 39 \\ 1 & 0 & 20 & 0 \\ 20 & 20 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{ccccccc} r^{\mathrm{s}} & r^{\mathrm{n}} & r^{\mathrm{fb}} & r^{\mathrm{ml}} & r^{\mathrm{rp}} & r^{\mathrm{b}} & r^{\mathrm{rb}} \\ 0.732 & 0.140 & 0.331 & -1.107 & 0.416 & 1.000 & 0.119 \\ 0.500 & 0.256 & 0.331 & -1.107 & 1.398 & 1.011 & 0.119 \\ 0.500 & 0.308 & 0.331 & -1.107 & 1.170 & 1.116 & 0.119 \\ 0.025 & 0.018 & 0.009 & -4.771 & -2.984 & 0.757 & -0.356 \end{array} \quad \begin{array}{c} A^1 \\ \begin{pmatrix} 0 & 1 & 20 \\ 1 & 0 & 20 \\ 20 & 20 & 0 \end{pmatrix} \end{array} \begin{array}{ccccc} r^{\mathrm{s}} & r^{\mathrm{n}} & r^{\mathrm{fb}} & r^{\mathrm{ml}} & r \\ 0.5 & 0.25 & 0.333 & -1.099 & \\ 0.5 & 0.25 & 0.333 & -1.099 & \\ 0.5 & 0.25 & 0.333 & -1.099 & \end{array}$$

From Figure 7.3.1, it is easily seen that Player 1 is a bridge player in A with $N^1 = \{1, 2, 3\}$ and $N^2 = \{1, 4\}$. In tournament A^1, all players are tied according to all ranking methods. Yet, in A, players 1 and 3 are not tied anymore according to φ^{s}, φ^{n}, φ^{rp}, and φ^{b}. Hence, these rules do not satisfy SS. ◁

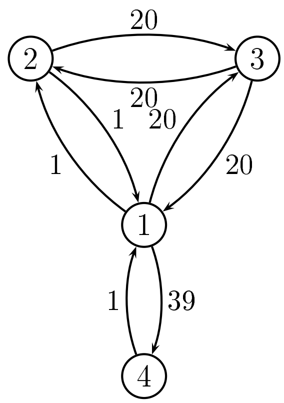

Figure 7.3.1: Graph representation of tournament A of Example 7.3.9

7.4 Response to victories and losses

In this section we consider two types of properties for a ranking method φ. The first type deals with preserving a ranking when two tournaments (N, A) and (N, A') are combined. The second type deals with the symmetric/asymmetric role victories and losses play in a ranking method.

Definition 7.4.1 The ranking method φ satisfies *flatness preservation* (FP) if for each pair of tournaments (N, A) and (N, A') it holds that $\varphi(A + A')$ is flat if both $\varphi(A)$ and $\varphi(A')$ are flat.

Definition 7.4.2 The ranking method φ satisfies *symmetry between victories and losses* (SVL) if for each tournament (N, A) and every $i, j \in N$, $i \; R^{\varphi(A)} \; j$ if and only if $j \; R^{\varphi(A^\top)} \; i$.

If we reverse all the results in a tournament, then the ranking should be reversed as well. In an example, Borm et al. (2004) discuss this property for the lambda ranking method. Note that SVL trivially implies SYM.

Definition 7.4.3 Define, for every $\lambda \in \mathbb{R}^N_{++}$, $\Lambda = \mathrm{diag}((\lambda_i)_{i\in N})$. The ranking method φ satisfies *negative response to losses* (NRL) if, for each (N, A) such that $\varphi(A)$ is flat and every $\lambda \in \mathbb{R}^N_{++}$, $i \; R^{\varphi(A\Lambda)} \; j$ if and only if $\lambda_i \leq \lambda_j$.

This property is introduced in Slutzki and Volij (2005) and is the key ingredient of the characterization they obtain for the fair bets ranking method. In words of the authors: "Negative responsiveness to losses concerns situations in which all players are equally ranked and the problem is irreducible. If a new problem is obtained by multiplying each player's losses by some positive constant (which may be different for each player), then the players should be ranked in the new problem in a way that is inversely related to these constants".

Theorem 7.4.4 φ^{s} satisfies FP.

Proof: Let (N, A) and (N, A') be tournaments such that $\varphi^{\mathrm{s}}(A)$ and $\varphi^{\mathrm{s}}(A')$ are flat. Let $i \in N$. Since $r^s_i(A) = \frac{1}{2}$ and $r^s_i(A') = \frac{1}{2}$ we have $r^s_i(A+A') = \frac{m_i r^{\mathrm{s}}_i(A) + m'_i r^{\mathrm{s}}_i(A')}{m_i + m'_i} = \frac{1}{2}$. Hence, $\varphi^s(A + A')$ is flat. □

The following theorem relates flatness of three ranking methods to flatness of φ^{s}.

Theorem 7.4.5 Let (N, A) be a tournament. Then for all $\varphi \in \{\varphi^{\mathrm{rp}}, \varphi^{\mathrm{rb}}, \varphi^{\mathrm{ml}}\}$ it holds that $\varphi(A)$ is flat if and only if $\varphi^{\mathrm{s}}(A)$ is flat.

Proof: We start with the proof for φ^{rp}. For the 'only if' part, assume that $\varphi^{\mathrm{rp}}(A)$ is flat, so $r^{\mathrm{rp}} = 0$. Recall that r^{rp} is a solution of $(I - \bar{M})r^{\mathrm{rp}} = \hat{c}$, where $\hat{c}_i$ is increasing on r^{s}_i. Then, $(I - \bar{M})r^{\mathrm{rp}} = 0$. Hence, $\hat{c} = 0$ and therefore, $\varphi^{\mathrm{s}}(A)$ is flat.
For the 'if' part, assume that $\varphi^{\mathrm{s}}(A)$ is flat, so $r^{\mathrm{s}} = \frac{1}{2}e$. Then, $c = 0$ and $\hat{c} = 0$. So,

a particular solution of $(I-\bar{M})x=\hat{c}$ is 0. Hence, $\varphi^{\mathrm{rp}}(A)$ is flat.

Now consider φ^{rb}. For the 'only if' part, assume that $\varphi^{\mathrm{rb}}(A)$ is flat, so $r^{\mathrm{rb}}=0$. Recall that r^{rb} is a solution of $(I-\bar{M})r^{\mathrm{rb}}=\hat{r^{\mathrm{s}}}$, where $\hat{r^{\mathrm{s}}}_i$ is increasing on r^{s}_i. Then, $(I-\bar{M})r^{\mathrm{rb}}=0$. Hence, $\hat{r^{\mathrm{s}}}=0$ and therefore, $\varphi^{\mathrm{s}}(A)$ is flat.
For the 'if' part, assume that $\varphi^{\mathrm{s}}(A)$ is flat, so $r^{\mathrm{s}}=\frac{1}{2}e$. Then, $\hat{r^{\mathrm{s}}}=0$. So, a particular solution of $(I-\bar{M})x=\hat{r^{\mathrm{s}}}$ is 0. Hence, $\varphi^{\mathrm{rb}}(A)$ is flat.

Lastly, consider φ^{ml}. For the 'only if' part, let (N,A) be such that $\varphi^{\mathrm{ml}}(A)$ is flat, so $\pi=\frac{1}{|N|}e$. Recall that for $i\in N$,

$$\pi_i=\frac{m_i r^{\mathrm{s}}_i}{\sum_{j\in N\setminus\{i\}}\frac{M_{ij}}{\pi_i+\pi_j}}.$$

Hence,

$$\begin{aligned} r^{\mathrm{s}}_i &= \frac{\frac{1}{|N|}\sum_{j\in N\setminus\{i\}}\frac{M_{ij}}{\frac{2}{|N|}}}{m_i}\\ &= \frac{\frac{1}{2}\sum_{j\in N\setminus\{i\}}M_{ij}}{m_i}\\ &= \frac{1}{2}. \end{aligned}$$

Hence, $r^{\mathrm{s}}_i=\frac{1}{2}$ for every $i\in N$ and therefore $\varphi^{\mathrm{s}}(A)$ is flat.

For the 'if' part, assume that $\varphi^{\mathrm{s}}(A)$ is flat, so $r^{\mathrm{s}}=\frac{1}{2}e$. From the previous part we obtain that $\pi_i=\frac{1}{|N|}$ for every $i\in N$ is a solution to

$$\pi_i=\frac{m_i r^{\mathrm{s}}_i}{\sum_{j\in N\setminus\{i\}}\frac{M_{ij}}{\pi_i+\pi_j}}.$$

Also, $\sum_{i\in N}\pi_i=1$. Hence, $\varphi^{\mathrm{ml}}(A)$ is flat. □

By FP of φ^{s} we obtain the following corollary.

Corollary 7.4.6 *φ^{rp}, φ^{rb} and φ^{ml} satisfy* FP.

Slutzki and Volij (2005) shows that φ^{fb} satisfies FP on a domain that differs from ours, the extension to our domain is straightforward.

Theorem 7.4.7 φ^{fb} satisfies FP

Proof: Let (N, A) and (N, A') be tournaments such that $\varphi^{\text{fb}}(A)$ and $\varphi^{\text{fb}}(A')$ are flat. This means that $r_i^{\text{fb}}(A) = r_i^{\text{fb}}(A') = \frac{1}{|N|}$ for every $i \in N$. Take $x_i = \frac{1}{|N|}$ for every $i \in N$. Clearly, $\sum_{i \in N} x_i = 1$. Also,

$$\begin{aligned}
\sum_{j \in N} (A + A')_{ij} x_j &= \sum_{j \in N} A_{ij} \frac{1}{|N|} + \sum_{j \in N} A'_{ij} \frac{1}{|N|} \\
&= \sum_{j \in N} A_{ij} r^{\text{fb}}(A) + \sum_{j \in N} A'_{ij} r^{\text{fb}}(A') \\
&= \sum_{j \in N} A_{ji} r^{\text{fb}}(A) + \sum_{j \in N} A'_{ji} r^{\text{fb}}(A') \\
&= \sum_{j \in N} (A + A')_{ji} \frac{1}{|N|} \\
&= \sum_{j \in N} (A + A')_{ji} x_i.
\end{aligned}$$

Therefore, $r^{\text{fb}}(A + A') = x$, so $\varphi^{\text{fb}}(A + A')$ is flat. □

The following example shows that φ^{n} and φ^{b} do not satisfy FP.

Example 7.4.8 Consider the tournaments A and A' described below:

$$\begin{array}{cc}
A & \begin{array}{cc} r^{\text{n}} & r^{\text{b}} \end{array} \\
\begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} & \begin{array}{cc} 0.222 & 1 \\ 0.222 & 1 \\ 0.222 & 1 \\ 0.222 & 1 \end{array}
\end{array}
\quad
\begin{array}{cc}
A' & \begin{array}{cc} r^{\text{n}} & r^{\text{b}} \end{array} \\
\begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} & \begin{array}{cc} 0.25 & 1 \\ 0.25 & 1 \\ 0.25 & 1 \\ 0.25 & 1 \end{array}
\end{array}
\quad
\begin{array}{cc}
A + A' & \begin{array}{cc} r^{\text{n}} & r^{\text{b}} \end{array} \\
\begin{pmatrix} 0 & 1 & 4 & 3 \\ 1 & 0 & 4 & 3 \\ 3 & 3 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix} & \begin{array}{cc} 0.255 & 1.021 \\ 0.255 & 1.021 \\ 0.240 & 0.990 \\ 0.227 & 0.966 \end{array}
\end{array}$$

Then $\varphi^{\text{n}}(A)$, $\varphi^{\text{n}}(A')$, $\varphi^{\text{b}}(A)$ and $\varphi^{\text{b}}(A')$ are all flat, but $\varphi^{\text{n}}(A+A')$ and $\varphi^{\text{b}}(A+A')$ are not. ◁

For the score ranking method φ^{s}, SVL is trivial.

Theorem 7.4.9 φ^{s} satisfies SVL.

For various other ranking methods, SVL can be shown by explicitly transforming the rating vector.

Theorem 7.4.10 φ^{ml} satisfies SVL.

Proof: Let (N, A) be a tournament. Recall that φ^{ml} orders the players according to $r^{\mathrm{ml}}(A)$ where, for each $i \in N$, $r^{\mathrm{ml}}(A)_i = \log(\pi(A)_i)$ and vector $\pi(A)$ is such that $(\pi(A))^\top e^N = 1$ and, for each $i \in N$,

$$\pi(A)_i = \frac{m_i r_i^{\mathrm{s}}(A)}{\sum_{j \in N\setminus\{i\}} \frac{M_{ij}}{\pi(A)_i + \pi(A)_j}}.$$

Let $\bar{x}$ be defined, for each $i \in N$, by $\bar{x}_i = r_i^{\mathrm{ml}} - \frac{1}{|N|}\sum_{j\in N} r_j^{\mathrm{ml}}(A)$. Then, $\sum_{i \in N} \bar{x}_i = 0$ and, for each $i \in N$, $\pi(A)_i = \alpha \exp(\bar{x}_i)$ with $\alpha = (\prod_{j\in N}\pi(A)_j)^{1/|N|}$ since

$$\begin{aligned}
\alpha \exp(\bar{x}_i) &= (\prod_{j\in N} \pi_j(A))^{1/|N|} \exp\left(r_i^{\mathrm{ml}}(A) - \frac{1}{|N|}\sum_{j\in N} r_j^{\mathrm{ml}}(A)\right) \\
&= (\prod_{j\in N} \pi_j(A))^{1/|N|} \exp\left(\log(\pi_i(A)) - \frac{1}{|N|}\sum_{j\in N} \log(\pi_j(A))\right) \\
&= (\prod_{j\in N} \pi_j(A))^{1/|N|} \frac{\pi_i(A)}{(\prod_{j\in N}\pi_j(A))^{1/|N|}} \\
&= \pi_i.
\end{aligned}$$

Hence, for each $i \in N$, we have

$$\exp(\bar{x}_i) = \frac{m_i r_i^{\mathrm{s}}(A)}{\sum_{j\in N\setminus\{i\}} M_{ij}\frac{1}{\exp(\bar{x}_i)+\exp(\bar{x}_j)}}. \tag{7.1}$$

Now consider the following system of equations in $y \in \mathbb{R}^N$: $\sum_{i\in N} y_i = 0$ and, for each $i \in N$,

$$\exp(y_i) = \frac{m_i(1 - r_i^{\mathrm{s}}(A))}{\sum_{j\in N\setminus\{i\}} M_{ij}\frac{1}{\exp(y_i)+\exp(y_j)}}. \tag{7.2}$$

If we show that $y = -\bar{x}$ solves this system, then (because the transformation from π to $\bar{x}$ is monotonic) we are done since player i's score in $A^\top$ is $1 - r_i^{\mathrm{s}}(A)$ and $M(A) = M(A^\top)$. Filling in $y = -\bar{x}$ in the right hand side of (7.2) yields

$$
\begin{aligned}
\frac{m_i(1-r_i^{\mathrm{s}}(A))}{\sum_{j\in N\setminus\{i\}} M_{ij}\frac{1}{\exp(-\bar{x}_i)+\exp(-\bar{x}_j)}} &= \frac{m_i(1-r_i^{\mathrm{s}}(A))}{\sum_{j\in N\setminus\{i\}} M_{ij}\frac{\exp(\bar{x}_i)\exp(\bar{x}_j)}{\exp(\bar{x}_i)+\exp(\bar{x}_j)}} \\
&= \frac{1}{\exp(\bar{x}_i)}\frac{m_i(1-r_i^{\mathrm{s}}(A))}{\sum_{j\in N\setminus\{i\}} M_{ij}\frac{\exp(\bar{x}_j)}{\exp(\bar{x}_i)+\exp(\bar{x}_j)}} \\
&= \frac{1}{\exp(\bar{x}_i)}\frac{m_i(1-r_i^{\mathrm{s}}(A))}{\sum_{j\in N\setminus\{i\}} M_{ij}(1-\frac{\exp(\bar{x}_i)}{\exp(\bar{x}_i)+\exp(\bar{x}_j)})} \\
&= \frac{1}{\exp(\bar{x}_i)}\frac{m_i(1-r_i^{\mathrm{s}}(A))}{m_i-\exp(\bar{x}_i)\sum_{j\in N\setminus\{i\}}\frac{M_{ij}}{\exp(\bar{x}_i)+\exp(\bar{x}_j)}},
\end{aligned}
$$

which, by (7.1), reduces to

$$
\frac{1}{\exp(\bar{x}_i)}\frac{m_i(1-r_i^{\mathrm{s}}(A))}{m_i-\exp(\bar{x}_i)\frac{m_i r_i^{\mathrm{s}}(A)}{\exp(\bar{x}_i)}} = \frac{1}{\exp(\bar{x}_i)} = \exp(y_i).
$$

So $y=-\bar{x}$ solves the system for $A^\top$ and therefore, φ^{ml} satisfies SVL. □

Theorem 7.4.11 φ^{rp}, φ^{rb}, and φ^{b} satisfy SVL.

Proof: Let (N,A) be a tournament. To show that φ^{rp} satisfies SVL, observe that if r^{rp} solves $\bar{M}x+\hat{c}(A)=x$, then $-r^{\mathrm{rp}}$ solves the corresponding equation for $A^\top$, because $\bar{M}=\bar{M}^\top$ and $\hat{c}(A^\top)=-\hat{c}(A)$ as a result of F^{-1} being symmetric around $\frac{1}{2}$. The argument for φ^{rb} is analogous. For φ^{b}, observe that $r^{\mathrm{s}}(A^\top)=e^N-r^{\mathrm{s}}(A)$, from which it readily follows that $r^{\mathrm{b}}(A^\top)=2e^N-(\bar{M}r^{\mathrm{s}}(A)+r^{\mathrm{s}}(A))=2e^N-r^{\mathrm{b}}(A)$ and so φ^{b} satisfies SVL as well. □

Not all ranking methods satisfy SVL, as is shown in de following example.

Example 7.4.12 Consider the following tournaments A and $A^\top$

$$
\begin{array}{c}A\\ \begin{pmatrix}0&0.5&0.2&1\\0.5&0&0.3&0.8\\0.8&0.7&0&0.9\\0&0.2&0.1&0\end{pmatrix}\end{array}
\begin{array}{cc}r^{\mathrm{n}}&r^{\mathrm{fb}}\\0.1756&0.195\\0.2011&0.210\\0.3056&0.559\\0.0622&0.036\end{array}
\quad\text{and}\quad
\begin{array}{c}A^\top\\ \begin{pmatrix}0&0.5&0.8&0\\0.5&0&0.7&0.2\\0.2&0.3&0&0.1\\1&0.8&0.9&0\end{pmatrix}\end{array}
\begin{array}{cc}r^{\mathrm{n}}&r^{\mathrm{fb}}\\0.1311&0.066\\0.1789&0.137\\0.1056&0.054\\0.3289&0.744\end{array}.
$$

Since player 2 is ranked above player 1 in both A and $A^\top$ and for both φ^{n} and φ^{fb}, these ranking methods do not satisfy SVL. ◁

Our analysis of NRL builds upon Slutzki and Volij (2005), though some care is needed as they develop their characterization of φ^{fb} for a class that allows for reducible tournaments, but is restricted to tournament matrices with integer entries.

A tournament A is called *balanced* if $Ae^N = A^\top e^N$, *i.e.*, if for every player, the total result against all his opponents equals the total result of all his opponents against him. It is strongly balanced if, moreover, there is a constant k such that $Ae^N = ke^N$, so the total result of a player against all other players (and the total result of all players against one player) is equal across all players. The next result is an adaptation of Lemmas 3 and 4 in Slutzki and Volij (2005).

Lemma 7.4.13 *Let φ be a ranking method satisfying* ANO*,* HOM*,* SYM *and* FP*. Then φ is flat on balanced tournaments.*

Proof: First, suppose that A is strongly balanced with $Ae = ke$. Then by Birkhoff's theorem (Birkhoff (1946)), matrix A can be written as k times a convex combination of permutation matrices. By ANO, φ is flat on permutation matrices. By HOM, φ is also flat on the tournaments that result after the multiplication of the permutation matrices by positive numbers. Finally, by FP and HOM again, φ is flat also on matrix A.

If A is not strongly balanced, then A can be decomposed as the sum of a strongly balanced tournament, in which we have just seen that φ is flat, and a symmetric tournament (see the proof of Lemma 4 in Slutzki and Volij (2005)). By SYM, φ is flat on the symmetric tournament as well, and by FP it is then flat on the original tournament A. □

Most of the ranking methods we consider in this chapter satisfy ANO, HOM, SYM, and FP, and, therefore, these ranking methods coincide (and are flat) for balanced tournaments. Slutzki and Volij (2005) provides a characterization of FP on the domain of all irreducible tournaments (N, A) such that A_{ij} is integer for every $i, j \in N$, $i \neq j$. The next result, which adapts this result to our setting, illustrates the strength of the NRL property.

Theorem 7.4.14 The fair bets ranking method, φ^{fb}, is the unique ranking method satisfying ANO, HOM, SYM, FP, and NRL.

Proof: φ^{fb} was already shown to satisfy ANO, HOM, SYM and FP. Regarding NRL, it is straightforward to extend the proof of Slutzki and Volij (2005) to our domain.

To show the converse, let φ be a ranking method satisfying ANO, HOM, SYM, FP, and NRL. Given an irreducible tournament A and corresponding fair bets rating vector, $r^{\mathrm{fb}}(A)$, the tournament $A' = A\,\mathrm{diag}((r_i^{\mathrm{fb}})_{i\in N})$ is a balanced (and irreducible) tournament because, by definition, for all $i \in N$,

$$\sum_{j\in N} A_{ij} r_j^{\mathrm{fb}}(A) = \sum_{j\in N} A_{ji} r_i^{\mathrm{fb}}(A).$$

Then, $A = A'(\mathrm{diag}((r_i^{\mathrm{fb}}(A))_{i\in N}))^{-1}$. Since φ satisfies ANO, HOM, SYM, and FP, by Lemma 7.4.13, $\varphi(A')$ is flat. Then, by NRL, $i\ R^{\varphi(A)}\ j$ if and only if $1/r_i^{\mathrm{fb}}(A) \leq 1/r_j^{\mathrm{fb}}(A)$, so $i\ R^{\varphi(A)}\ j$ if and only if $i\ R^{\varphi^{\mathrm{fb}}(A)}\ j$. □

As a result of Theorem 7.4.14, φ^{s}, φ^{ml}, φ^{rp} and φ^{rb} do not satisfy NRL because they satisfy all other properties in the characterization. The following example shows that φ^{n} and φ^{b} do not satisfy NRL either.

Example 7.4.15 Let $\lambda = (0.99, 2, 1, 1)$ and $\Lambda = \mathrm{diag}((\lambda_i)_{i\in N})$. Let A and $A\Lambda$ be as follows:

$$\begin{array}{cc} A & \begin{array}{cc} r^{\mathrm{n}} & r^{\mathrm{b}} \end{array} \\ \begin{pmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix} & \begin{array}{cc} 0.25 & 1 \\ 0.25 & 1 \\ 0.25 & 1 \\ 0.25 & 1 \end{array} \end{array} \quad \text{and} \quad \begin{array}{cc} A\Lambda & \begin{array}{cc} r^{\mathrm{n}} & r^{\mathrm{b}} \end{array} \\ \begin{pmatrix} 0 & 4 & 1 & 1 \\ 1.98 & 0 & 1 & 1 \\ 0.99 & 2 & 0 & 2 \\ 0.99 & 2 & 2 & 0 \end{pmatrix} & \begin{array}{cc} 0.245 & 1.024 \\ 0.192 & 0.911 \\ 0.264 & 1.046 \\ 0.264 & 1.046 \end{array} \end{array}$$

Note that both φ^{n} and φ^{b} are flat on A but, despite $\lambda_1 \leq \lambda_3$, $r_3^{\mathrm{n}}(A\Lambda) > r_1^{\mathrm{n}}(A\Lambda)$ and $r_3^{\mathrm{b}}(A\Lambda) > r_1^{\mathrm{b}}(A\Lambda)$. ◁

7.5 Score consistency

In this section we investigate to what extent the various ranking methods preserve some of the features of the scores ranking method that make it quite appealing for round-robin tournaments.

Definition 7.5.1 The ranking method φ satisfies *score consistency* (SCC) if $\varphi(A) = \varphi^{\mathrm{s}}(A)$ for every round-robin tournament (N, A).

Definition 7.5.2 The ranking method φ satisfies *strong score consistency* (SSCC) if, for all (N, A) and all $i, j \in N$ such that $M_{ik} = M_{jk}$ for all $k \in N \backslash \{i, j\}$, $i \; R^{\varphi(A)} \; j$ if and only if $i \; R^{\varphi^{\mathrm{s}}(A)} \; j$.

If i and j play the same amount of matches against the other players, then they should be ranked according to their scores. Note that SSCC trivially implies SCC.

Example 7.5.3 The tournament A of Example 7.4.12 shows that φ^{n} and φ^{fb} do not satisfy SCC.

$$\begin{array}{cccc}
A & r^{\mathrm{s}} & r^{\mathrm{n}} & r^{\mathrm{fb}} \\
\begin{pmatrix} 0 & 0.5 & 0.2 & 1 \\ 0.5 & 0 & 0.3 & 0.8 \\ 0.8 & 0.7 & 0 & 0.9 \\ 0 & 0.2 & 0.1 & 0 \end{pmatrix} & \begin{array}{c} 0.567 \\ 0.533 \\ 0.800 \\ 0.100 \end{array} & \begin{array}{c} 0.176 \\ 0.201 \\ 0.306 \\ 0.062 \end{array} & \begin{array}{c} 0.195 \\ 0.210 \\ 0.559 \\ 0.036 \end{array}
\end{array}$$

It is readily checked that A is round-robin. Since φ^{s} ranks player 1 above player 2 and both φ^{n} and φ^{fb} rank player 2 above player 1, φ^{n} and φ^{fb} do not satisfy SCC. $\triangleleft$

The remaining ranking methods all satisfy SSCC, and hence SCC.

Theorem 7.5.4 φ^{ml} satisfies SSCC.

Proof: Let (N, A) and $i, j \in N$ be tournaments such that $M_{ik} = M_{jk}$ for all $k \in N \backslash \{i, j\}$. Using the definition of φ^{ml} we have

$$r_i^{\mathrm{s}}(A) = \frac{1}{m_i} \sum_{k \in N \backslash \{i\}} M_{ik} \frac{\pi_i(A)}{\pi_i(A) + \pi_k(A)}.$$

Since $\frac{\pi_i(A)}{\pi_i(A)+\pi_k(A)}$ is increasing in $\pi_i(A)$, the right hand side of the equation is increasing in $\pi_i(A)$. Then, because $M_{ik} = M_{jk}$ for all $k \neq i, j$ and therefore $m_i = m_j$, we have that $r_i^{\mathrm{s}}(A) \geq r_j^{\mathrm{s}}(A)$ if and only if $\pi_i(A) \geq \pi_j(A)$. Hence, φ^{ml} satisfies SSCC. $\Box$

If $|N| = 2$ we have that $Mr^{\mathrm{s}} + r^{\mathrm{s}} = (r_1^{\mathrm{s}} + r_2^{\mathrm{s}}, r_1^{\mathrm{s}} + r_2^{\mathrm{s}})$, so φ^{b} is flat in two-player tournaments and therefore satisfies neither SSCC nor SCC.

Theorem 7.5.5 If $|N| > 2$, then φ^{b} satisfies SSCC.

Proof: Let (N, A) be a tournament and $i, j \in N$ be such that $M_{ik} = M_{jk}$ for all $k \in N \backslash \{i, j\}$. Given $i, j \in N$, since $M_{ik} = M_{jk}$ for all $k \neq i, j$, we have that $m_i = m_j$ and, hence, $\bar{M}_{ij} = \bar{M}_{ji}$. Then,

$$\begin{aligned}
r_i^{\mathrm{b}} - r_j^{\mathrm{b}} &= (\bar{M}r^{\mathrm{s}}(A) + r^{\mathrm{s}}(A))_i - (\bar{M}r^{\mathrm{s}}(A) + r^{\mathrm{s}}(A))_j \\
&= \sum_{r\in N} \frac{M_{ir}}{m_i} r_r^{\mathrm{s}}(A) + r_i^{\mathrm{s}}(A) - \sum_{r\in N} \frac{M_{jr}}{m_j} r_r^{\mathrm{s}}(A) - r_j^{\mathrm{s}}(A) \\
&= \frac{M_{ij}}{m_i} r_j^{\mathrm{s}}(A) + r_i^{\mathrm{s}}(A) - \frac{M_{ji}}{m_j} r_i^{\mathrm{s}}(A) - r_j^{\mathrm{s}}(A) \\
&= (1 - \bar{M}_{ij})(r_i^{\mathrm{s}}(A) - r_j^{\mathrm{s}}(A)).
\end{aligned}$$

Since A is irreducible and $|N| > 2$, it cannot be the case that $\bar{M}_{ij} = 1$. Then, $(1 - \bar{M}_{ij}) > 0$ and φ^{b} and φ^{s} produce the same ranking. □

Theorem 7.5.6 φ^{rb} and φ^{rp} satisfy SSCC.

Proof: Let (N, A) be a tournament and $i, j \in N$ be such that $M_{ik} = M_{jk}$ for all $k \in N\backslash\{i,j\}$. Recall that φ^{rb} solves $(I - \bar{M})x = \hat{r^{\mathrm{s}}}(A)$. So, in particular

$$x_i - \bar{M}_{ij}x_j - \sum_{k\in N\backslash\{i,j\}} \bar{M}_{ik}x_k = \hat{r^{\mathrm{s}}}_i(A),$$

and

$$-\bar{M}_{ji}x_i + x_j - \sum_{k\in N\backslash\{i,j\}} \bar{M}_{jk}x_k = \hat{r^{\mathrm{s}}}_j(A).$$

Substracting the two equations and using that $\bar{M}_{ij} = \bar{M}_{ji}$ and $\bar{M}_{ik} = \bar{M}_{jk}$ for all other k yields

$$(1 + \bar{M}_{ij})(x_i - x_j) = \hat{r^{\mathrm{s}}}_i(A) - \hat{r^{\mathrm{s}}}_j(A).$$

Therefore, $x_i - x_j$ and $\hat{r^{\mathrm{s}}}_i(A) - \hat{r^{\mathrm{s}}}_j(A)$ have the same sign. Hence, φ^{rb} satisfies SSCC. The proof for φ^{rp} is analogous, but with $\hat{c}(A)$ in the right hand side. Since $\hat{c}(A)$ and $\hat{r^{\mathrm{s}}}(A)$ induce the same ranking, the same argument works. □

7.6 Monotonicity

In this section we present three properties that deal with changes in the tournament matrix. If an existing result is changed or a new one is added, how should the rankings change? For independence of irrelevant matches, we use the definition by Rubinstein (1980)

Definition 7.6.1 The ranking method φ satisfies *independence of irrelevant matches* (IIM) if, for each (N, A), each $k, \ell \in N$ and each (N, A') an identical tournament to (N, A) except for the results between k and ℓ, $i \ R^{\varphi(A)} \ j$ if and only if $i \ R^{\varphi(A')} \ j$ for every $i, j \in N \backslash \{k, \ell\}$.

This property states that for any two players, their relative ranking is not influenced by a change in a result involving neither of them.

Definition 7.6.2 The ranking method φ satisfies *positive responsiveness to the beating relation* (PRB) if, for each (N, A), each $i, k \in N$, $i \neq k$, each tournament (N, A') identical to (N, A) except that $A'_{ik} + A'_{ki} = A_{ik} + A_{ki}$ and $A'_{ik} > A_{ik}$, and for every $j \in N \backslash \{i\}$, $i \ P^{\varphi(A)} \ j$ if $i \ R^{\varphi(A)} \ j$. Note that this should hold in particular for $k = j$.

Definition 7.6.3 The ranking method φ satisfies *non-negative responsiveness to the beating relation* (NNRB) if, for each (N, A), each $i, k \in N$, $i \neq k$, each tournament (N, A') identical to (N, A) except that $A'_{ik} + A'_{ki} = A_{ik} + A_{ki}$ and $A'_{ik} > A_{ik}$, and for every $j \in N \backslash \{i\}$ the following holds: if $i \ R^{\varphi(A)} \ j$ then $i \ R^{\varphi(A')} \ j$ and, if $k = j$, $i \ P^{\varphi(A')} \ j$.

These two properties state that an improved result should always be beneficial to your ranking. Of course, PRB trivially implies NNRB. Trivially, the scores ranking method satisfies IIM and PRB, and therefore NNRB.

Theorem 7.6.4 φ^{s} satisfies IIM, PRB and NNRB.

We show below that all the other ranking methods violate IIM.

Example 7.6.5 Consider the tournaments A and A' described below:

$$\begin{array}{c} A \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array} \quad \begin{array}{cccccc} r^{\mathrm{n}} & r^{\mathrm{fb}} & r^{\mathrm{ml}} & r^{\mathrm{rp}} & r^{\mathrm{b}} & r^{\mathrm{rb}} \\ 0.224 & 0.2 & -1.609 & -0.201 & 0.959 & -0.05 \\ 0.265 & 0.3 & -1.204 & 0.201 & 1.041 & 0.05 \\ 0.265 & 0.3 & -1.204 & 0.201 & 1.041 & 0.05 \\ 0.224 & 0.2 & -1.609 & -0.201 & 0.959 & -0.05 \end{array}$$

$$\begin{array}{c} A' \\ \begin{pmatrix} 0 & 1 & 1 & 2 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array} \quad \begin{array}{cccccc} r^{\mathrm{n}} & r^{\mathrm{fb}} & r^{\mathrm{ml}} & r^{\mathrm{rp}} & r^{\mathrm{b}} & r^{\mathrm{rb}} \\ 0.237 & 0.233 & -1.430 & -0.015 & 0.998 & -0.004 \\ 0.278 & 0.308 & -1.189 & 0.224 & 1.056 & 0.055 \\ 0.260 & 0.292 & -1.230 & 0.182 & 1.038 & 0.045 \\ 0.205 & 0.167 & -1.807 & -0.390 & 0.920 & -0.096 \end{array}$$

In tournament A, all ranking methods rank players 2 and 3 equally. In tournament A', except for φ^{s}, all ranking methods rank player 2 on top of player 3, violating IIM. ◁

Note that every ranking method satisfying score consistency satisfies IIM on the domain of round-robin tournaments. The scores ranking method φ^{s} also turns out to be the only one satisfying PRB.

Example 7.6.6 Consider the tournaments A and A' described below:

$$\begin{array}{c} A \\ \begin{pmatrix} 0 & 1 & 20 & 20 \\ 1 & 0 & 20 & 0 \\ 20 & 20 & 0 & 0 \\ 20 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{ccccccc} r^{\mathrm{s}} & r^{\mathrm{n}} & r^{\mathrm{fb}} & r^{\mathrm{ml}} & r^{\mathrm{rp}} & r^{\mathrm{b}} & r^{\mathrm{rb}} \\ 0.5 & 0.25 & 0.25 & -1.386 & 0 & 1 & 0 \\ 0.5 & 0.25 & 0.25 & -1.386 & 0 & 1 & 0 \\ 0.5 & 0.25 & 0.25 & -1.386 & 0 & 1 & 0 \\ 0.5 & 0.25 & 0.25 & -1.386 & 0 & 1 & 0 \end{array}$$

$$\begin{array}{c} A' \\ \begin{pmatrix} 0 & 1 & 20 & 39 \\ 1 & 0 & 20 & 0 \\ 20 & 20 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{ccccccc} r^{\mathrm{s}} & r^{\mathrm{n}} & r^{\mathrm{fb}} & r^{\mathrm{ml}} & r^{\mathrm{rp}} & r^{\mathrm{b}} & r^{\mathrm{rb}} \\ 0.732 & 0.140 & 0.331 & -1.107 & 0.416 & 1.000 & 0.119 \\ 0.500 & 0.256 & 0.331 & -1.107 & 1.398 & 1.011 & 0.119 \\ 0.500 & 0.308 & 0.331 & -1.107 & 1.170 & 1.116 & 0.119 \\ 0.025 & 0.018 & 0.008 & -4.771 & -2.984 & 0.757 & -0.356 \end{array}$$

According to all methods, players 1, 2 and 3 are equally ranked in A. In A', player 1 has a better result against player 4 than in A, but only r^{s} ranks him above players 2 and 3. Hence, all of them but r^{s} violate PRB. Moreover, φ^{n}, φ^{rp} and φ^{b} actually rank player 1 lower than 2 and 3 in A', so these three methods do not satisfy NNRB either. ◁

In the remainder of this section we show that both φ^{fb} and φ^{rb} satisfy NNRB. Although we conjecture that φ^{ml} also satisfies NNRB, this is still an open question. The result for φ^{fb} below extends the result in Levchenkov (1992) for round-robin tournaments (we build upon the proof in Laslier (1997)). We start with an auxiliary result that will be crucial in the proof for both φ^{fb} and φ^{rb}.

Lemma 7.6.7 *Let $B \in \mathbb{R}^{n\times n}$ be such that*

(i) *B is invertible,*

(ii) *for all $i \neq j$, $B_{ij} \leq 0$, and*

(iii) *$\sum_{j=1}^{n} B_{ji} \geq 0$ for all $i \in N$.*

Then, for all $\lambda \in \mathbb{R}^n_+$, $B^{-1}\lambda \geq 0$.

Proof: First note that (i)-(iii) imply that $B_{ii} > 0$ for all i since invertibility precludes a zero column. We do the proof by induction on n, the size of the square matrix B. For $n = 1$, the result follows immediately from $B_{11} > 0$. If $n > 1$, suppose the result is true for matrices of size $n-1$ and let $\lambda \in \mathbb{R}^n_+$ and $\gamma = B^{-1}\lambda$. The last equation of $B\gamma = \lambda$ can be written as

$$B_{nn}\gamma_n = \lambda_n - \sum_{j=1}^{n-1} B_{nj}\gamma_j. \tag{7.3}$$

Now, we substitute γ_n in the other equations and, for each $i \in \{1, \ldots, n-1\}$, the equation $\sum_{j=1}^{n} B_{ij}\gamma_j = \lambda_i$ can be rewritten as

$$\sum_{j=1}^{n-1}(B_{nn}B_{ij} - B_{in}B_{nj})\gamma_j = B_{nn}\lambda_i - B_{in}\lambda_n.$$

Define $\bar{B} \in \mathbb{R}^{(n-1)\times(n-1)}$ by $\bar{B}_{ij} = B_{nn}B_{ij} - B_{in}B_{nj}$ for all $i, j \in \{1, \ldots, n-1\}$. Also, define $\bar{\gamma}, \bar{\lambda} \in \mathbb{R}^{n-1}$ by $\bar{\gamma}_i = \gamma_i$ and $\bar{\lambda}_i = B_{nn}\lambda_i - B_{in}\lambda_n$ for all $i \in \{1, \ldots, n-1\}$. Then the above $n-1$ equations can be expressed in matrix form as $\bar{B}\bar{\gamma} = \bar{\lambda}$. It is now easy to check that $\bar{\lambda} \geq 0$, $\bar{B}$ is invertible, and for all $i \neq j$, $\bar{B}_{ij} \leq 0$. Thus, in order to apply the induction hypothesis we just need to show that $\sum_{j=1}^{n-1} \bar{B}_{ji} \geq 0$:

$$\begin{aligned}
\sum_{j=1}^{n-1} \bar{B}_{ji} &= \bar{B}_{ii} + \sum_{\substack{j=1\\ j\neq i}}^{n-1} \bar{B}_{ji} \\
&= B_{nn}B_{ii} - B_{in}B_{ni} + \sum_{\substack{j=1\\ j\neq i}}^{n-1}(B_{nn}B_{ji} - B_{jn}B_{ni}) \\
&= B_{nn}(B_{ii} + \sum_{\substack{j=1\\ j\neq i}}^{n-1} B_{ji}) - B_{ni}(B_{in} + \sum_{\substack{j=1\\ j\neq i}}^{n-1} B_{jn}) \\
&= B_{nn}(\sum_{j=1}^{n-1} B_{ji}) - B_{ni}(\sum_{j=1}^{n-1} B_{jn}) \\
&\overset{(iii)}{\geq} B_{nn}(-B_{ni}) - B_{ni}(-B_{nn}) \\
&= 0.
\end{aligned}$$

Therefore, we can apply the induction hypothesis to conclude that $\bar{\gamma}$ is nonnegative. Finally, nonnegativity of γ_n easily follows from (7.3), $B_{nn} > 0$ and nonnegativity of the other components of γ. $\square$

Theorem 7.6.8 φ^{rb} satisfies NNRB.

Proof: Let (N, A) be a tournament, and let $i, k \in N$, $i \neq k$. Below we explicitly characterize how the recursive Buchholz ranking varies as a function of A_{ik} and A_{ki}, provided that M_{ik} stays constant. Recall that $r^{\mathrm{rb}}(A)$ is the unique solution of $\bar{M}x + \hat{r^{\mathrm{s}}}(A) = x$ such that $(r^{\mathrm{rb}}(A))^\top e = 0$. Hence, $(I - \bar{M})r^{\mathrm{rb}}(A) = \hat{r^{\mathrm{s}}}(A)$. Define $B = I - \bar{M}$ and $\breve{N} = N \backslash \{i, k\}$. Then, equation ℓ of the system $Br^{\mathrm{rb}} = \hat{r^{\mathrm{s}}}(A)$ can be written as

$$\sum_{h \in \breve{N}} B_{\ell h} r_h^{\mathrm{rb}}(A) = \hat{r^{\mathrm{s}}}_\ell(A) - B_{\ell i} r_i^{\mathrm{rb}}(A) - B_{\ell k} r_k^{\mathrm{rb}}(A). \tag{7.4}$$

with $\ell \in \breve{N}$. Define $\breve{B} \in \mathbb{R}^{(n-2)\times(n-2)}$ to be the matrix obtained from B by deleting the rows and columns corresponding to players i and k.

We prove now that $\breve{B}$ is invertible. Suppose, on the contrary, that there is an $y \in \mathbb{R}^{\breve{N}}$, $y \neq 0$, such that $y^\top \breve{B}^\top = 0$. Let $\ell \in \breve{N}$ be such that $y_\ell = \max_{h \in \breve{N}} y_h$. We assume, without loss of generality, that $y_\ell > 0$. For each $h \neq \ell$, $B^\top_{h\ell} \leq 0$ and, hence, $-y_h B^\top_{h\ell} \leq -y_\ell B^\top_{h\ell}$, with equality only if $y_h = y_\ell$ or $B^\top_{h\ell} = 0$. Since $y^\top \breve{B}^\top = 0$, $\sum_{h \in \breve{N} \backslash \{\ell\}} -y_h B^\top_{h\ell} = y_\ell B^\top_{\ell\ell}$. Further, since $\sum_{h \in N} B^\top_{h\ell} = 0$, we have $\sum_{h \in \breve{N} \backslash \{\ell\}} -B^\top_{h\ell} = B^\top_{\ell\ell} + B^\top_{i\ell} + B^\top_{k\ell} \leq B^\top_{\ell\ell}$, with equality only if $B^\top_{i\ell} = B^\top_{k\ell} = 0$. Then, we have

$$y_\ell B^\top_{\ell\ell} = \sum_{h \in \breve{N} \backslash \{\ell\}} -y_h B^\top_{h\ell} \leq y_\ell \sum_{h \in \breve{N} \backslash \{\ell\}} -B^\top_{h\ell} \leq y_\ell B^\top_{\ell\ell}$$

and, hence, all the inequalities are indeed equalities. Therefore, $B^\top_{i\ell} = B^\top_{k\ell} = 0$ and, for each $h \in \breve{N} \backslash \{\ell\}$, $y_h = y_\ell$ or $B^\top_{h\ell} = 0$. Define $\bar{N} = \{m \in \breve{N} \mid y_m = \max_{h \in \breve{N}} y_h\}$. Now, for each $m \in \bar{N}$, we have $B^\top_{im} = B^\top_{km} = 0$ and, further, for each $h \in \breve{N} \backslash \bar{N}$, $B^\top_{hm} = 0$. That is, no player outside $\bar{N}$ has played against players inside $\bar{N}$, which contradicts the irreducibility of A.

Define $C = (\breve{B})^{-1}$, $\breve{r}^{\mathrm{rb}} = (r_h^{\mathrm{rb}})_{h \in \breve{N}}$, $\breve{r}^{\mathrm{s}} = (r_h^{\mathrm{s}}(A))_{h \in \breve{N}}$, $B^i = (B_{hi})_{h \in \breve{N}}$ and $B^k = (B_{hk})_{h \in \breve{N}}$. Then, using (7.4) we have $\breve{B}\breve{r}^{\mathrm{rb}} = \breve{r}^{\mathrm{s}} - B^i r_i^{\mathrm{rb}} - B^k r_k^{\mathrm{rb}}$ and hence, $\breve{r}^{\mathrm{rb}} = C(\breve{r}^{\mathrm{s}} - B^i r_i^{\mathrm{rb}} - B^k r_k^{\mathrm{rb}})$. So, for all $\ell \in \breve{N}$,

$$r_\ell^{\mathrm{rb}} = \breve{r}_\ell^{\mathrm{rb}} = \sum_{h\in\breve{N}} C_{\ell h}(\breve{r^{\mathrm{s}}}_h - B_{hi}r_i^{\mathrm{rb}} - B_{hk}r_k^{\mathrm{rb}}).$$

Define $\gamma_\ell^s = \sum_{h\in\breve{N}} C_{\ell h}\breve{r^{\mathrm{s}}}_h$, $\gamma_\ell^i = -\sum_{h\in\breve{N}} C_{\ell h}B_{hi}$ and $\gamma_\ell^k = -\sum_{h\in\breve{N}} C_{\ell h}B_{hk}$. Then, for each $\ell \in \breve{N}$,

$$r_\ell^{\mathrm{rb}} = \gamma_\ell^s + \gamma_\ell^i r_i^{\mathrm{rb}}(A) + \gamma_\ell^k r_k^{\mathrm{rb}}(A). \tag{7.5}$$

Furthermore, equation i in $Br^{\mathrm{rb}} = \hat{r^{\mathrm{s}}}(A)$ is

$$B_{ii}r_i^{\mathrm{rb}}(A) + B_{ik}r_k^{\mathrm{rb}}(A) + \sum_{\ell\in\breve{N}} B_{i\ell}r_\ell^{\mathrm{rb}}(A) = \hat{r^{\mathrm{s}}}_i(A). \tag{7.6}$$

Define $\Gamma^{i,i} = -\sum_{\ell\in\breve{N}} B_{i\ell}\gamma_\ell^i$ and $\Gamma^{i,k} = -\sum_{\ell\in\breve{N}} B_{i\ell}\gamma_\ell^k$. Then, plugging in the expression of each $r_\ell^{\mathrm{rb}}(A)$ (7.5) into (7.6) we get

$$(B_{ii} - \Gamma^{i,i})r_i^{\mathrm{rb}}(A) + (B_{ik} - \Gamma^{i,k})r_k^{\mathrm{rb}}(A) = \hat{r^{\mathrm{s}}}_i - \sum_{\ell\in\breve{N}} \gamma_\ell^s. \tag{7.7}$$

Now, adding up (7.5) over all $\ell \in \breve{N}$ and using that $\sum_{h\in N} r_h^{\mathrm{rb}}(A) = 0$,

$$(1 + \sum_{\ell\in\breve{N}} \gamma_\ell^i)r_i^{\mathrm{rb}}(A) + (1 + \sum_{\ell\in\breve{N}} \gamma_\ell^k)r_k^{\mathrm{rb}}(A) = -\sum_{\ell\in\breve{N}} \gamma_\ell^s. \tag{7.8}$$

Define $\sigma_i = \sum_{\ell\in\breve{N}} \gamma_\ell^i$ and $\sigma_k = \sum_{\ell\in\breve{N}} \gamma_\ell^k$. Then, solving equations (7.7) and (7.8), we get

$$r_i^{\mathrm{rb}} = \frac{\hat{r^{\mathrm{s}}}_i(A) - (1 - \frac{B_{ik}-\Gamma^{i,k}}{1+\sigma_k})\sum_{\ell\in\breve{N}} \gamma_\ell^s}{(B_{ii} - \Gamma^{i,i}) - (B_{ik} - \Gamma^{i,k})\frac{1+\sigma_k}{1+\sigma_i}} \quad \text{and} \quad r_k^{\mathrm{rb}} = \frac{-\sum_{\ell\in\breve{N}} \gamma_\ell^s}{1+\sigma_k} - \frac{1+\sigma_i}{1+\sigma_k} r_i^{\mathrm{rb}}. \tag{7.9}$$

To understand how $r_i^{\mathrm{rb}}(A)$ and $r_k^{\mathrm{rb}}(A)$ vary with $\hat{r^{\mathrm{s}}}_i(A)$, it is convenient to know the signs of γ^i and γ^k. We claim that both γ^i and γ^k are nonnegative vectors. By definition, $\gamma^i = -CB^i$ and, since $C^{-1} = \breve{B}$, $\breve{B}\gamma = -B^i$. Furthermore, $-B^i \geq 0$. Since matrix $\breve{B}$ and vectors γ^i and $-B^i$ satisfy the conditions of Lemma 7.6.7, γ^i is nonnegative. The argument for γ^k is analogous using $-B^k$ instead of $-B^i$. The nonnegativity of γ^i and γ^k implies that σ_i and σ_k are also nonnegative. Since γ^k is nonnegative, also $\Gamma^{i,k}$ is nonnegative and $B_{ik} - \Gamma^{i,k}$ is negative. Furthermore,

$$B_{ii} - \Gamma^{i,i} = B_{ii} + \sum_{\ell\in\breve{N}} B_{i\ell}\gamma_\ell^i \geq B_{ii} + \sum_{\ell\in\breve{N}} B_{i\ell} \geq 0.$$

We reexamine now equation (7.9). Note that γ^s, γ^i, γ^k, $\Gamma^{i,i}$, $\Gamma^{i,k}$, B_{ii}, and B_{ik} only depend on $\breve{B}$. Then, the denominator of the expression for $r_i^{\mathrm{rb}}(A)$ is positive and so

$r_i^{\mathrm{rb}}(A)$ is strictly increasing in $\hat{r^{\mathrm{s}}}_i(A)$. Further, since $r_k^{\mathrm{rb}}(A)$ is strictly decreasing in $r_i^{\mathrm{rb}}(A)$, it is strictly decreasing in $\hat{r^{\mathrm{s}}}_i(A)$.
Now, because of (7.5), r_ℓ^{rb} is weakly increasing in $r_i^{\mathrm{rb}}(A)$ and $r_k^{\mathrm{rb}}(A)$. Yet, since $r_i^{\mathrm{rb}}(A)$ and $r_k^{\mathrm{rb}}(A)$ are strictly increasing and decreasing, respectively, in $\hat{r^{\mathrm{s}}}_i(A)$, some extra work is needed to understand how $r_\ell^{\mathrm{rb}}(A)$ varies with $\hat{r^{\mathrm{s}}}_i(A)$. To do so, we first show that all the components of γ^i and γ^k are no larger than 1. We prove it for γ^i, the proof for γ^k being analogous.

$$\breve{B}(e^N - \gamma^i) = \breve{B}e^N - \breve{B}\gamma^i = \breve{B}e^N - B^i,$$

and, for each $\ell \in \breve{N}$,

$$(\breve{B}e^N - B^i)_\ell = \sum_{h\in\breve{N}} B_{\ell h} + B_{\ell i} \geq \sum_{h\in\breve{N}} B_{\ell h} + B_{\ell i} + B_{ki} = 0.$$

Then, since $\breve{B}e^N - B^i$ is a nonnegative vector, matrix $\breve{B}$ and vectors $e^N - \gamma^i$ and $\breve{B}e^N - B^i$ are in the conditions of Lemma 7.6.7 and, hence, $e^N - \gamma^i$ is nonnegative. Therefore, we know that all the components of γ^i and γ^k are no larger than 1. Looking again at equation (7.5), we have that $r_\ell^{\mathrm{rb}}(A)$ cannot increase with $\hat{r^{\mathrm{s}}}_i(A)$ faster than $r_i^{\mathrm{rb}}(A)$ so $\frac{r_\ell^{\mathrm{rb}}(A)}{r_i^{\mathrm{rb}}(A)}$ is weakly decreasing in $\hat{r^{\mathrm{s}}}_i(A)$. Similarly, $r_\ell^{\mathrm{rb}}/r_k^{\mathrm{rb}}$ is weakly increasing in $\hat{r^{\mathrm{s}}}_i(A)$. From this, the statement follows. $\square$

Theorem 7.6.9 φ^{fb} satisfies NNRB.

Proof: Let (N, A) be a tournament, and let $i, k \in N$, $i \neq k$. Below we explicitly characterize how the fair bets ranking varies as a function of A_{ik} and A_{ki}, provided that M_{ik} stays constant. Recall that $r^{\mathrm{fb}}(A)$ is the unique solution of $L_A^{-1}Ax = x$ such that $(r^{\mathrm{fb}}(A))^\top e = 1$, where $L_A = \mathrm{diag}((\tilde{a}_i)_{i\in N})$ is the diagonal matrix in which $\tilde{a}_i = (A^\top e^N)_i$ is the total amount of losses of player $i \in N$. Then, $Ar^{\mathrm{fb}}(A) = L_A r^{\mathrm{fb}}(A)$ and so $(A - L_A)r^{\mathrm{fb}}(A) = 0$. Define $B = L_A - A$ and $\breve{N} = N \backslash \{i, k\}$. Then, equation ℓ of the system $Br^{\mathrm{fb}}(A) = 0$ can be written as

$$\sum_{h\in\breve{N}} B_{\ell h} r_h^{\mathrm{fb}}(A) = -B_{\ell i} r_i^{\mathrm{fb}}(A) - B_{\ell k} r_k^{\mathrm{fb}}(A), \tag{7.10}$$

with $\ell \in \breve{N}$. Define $\breve{B} \in \mathbb{R}^{\breve{N}\times\breve{N}}$ to be the matrix obtained from B by deleting the rows and columns corresponding to players i and k.

We prove now that $\breve{B}$ is invertible. Suppose, on the contrary, that there is an $y \in \mathbb{R}^{\breve{N}}, y \neq 0$ such that $y^\top \breve{B} = 0$. Let $\ell \in \breve{N}$ be such that $y_\ell = \max_{h\in\breve{N}} y_h$. We assume, without loss of generality, that $y_\ell > 0$. For each $h \in \breve{N}\backslash\{\ell\}$, $B_{h\ell} \leq 0$ and, hence, $-y_h B_{h\ell} \leq -y_\ell B_{h\ell}$, with equality only if $y_h = y_\ell$ or $B_{h\ell} = 0$. Since $y^\top \breve{B} = 0$, $\sum_{h\in\breve{N}\backslash\{\ell\}} -y_h B_{h\ell} = y_\ell B_{\ell\ell}$. Further, since $\sum_{h\in N} B_{h\ell} = 0$, we have $\sum_{h\in\breve{N}\backslash\{\ell\}} -B_{h\ell} = B_{\ell\ell} + B_{i\ell} + B_{k\ell} \leq B_{\ell\ell}$, with equality only if $B_{i\ell} = B_{k\ell} = 0$. Then, we have

$$y_\ell B_{\ell\ell} = \sum_{h\in\breve{N}\backslash\{\ell\}} -y_h B_{h\ell} \leq y_\ell \sum_{h\in\breve{N}\backslash\{\ell\}} -B_{h\ell} \leq y_\ell B_{\ell\ell}$$

and, hence, all the inequalities are indeed equalities. Therefore, $B_{i\ell} = B_{k\ell} = 0$ and, for each $h \in \breve{N}\backslash\{\ell\}$, $y_h = y_\ell$ or $B_{h\ell} = 0$. Define $\bar{N} = \{m \in \breve{N} \,|\, y_m = \max_{h\in\breve{N}} y_h\}$. Now, for each $m \in \bar{N}$, we have $B_{im} = B_{km} = 0$ and, further, for each $h \in \breve{N}\backslash\bar{N}$, $B_{hm} = 0$. That is, no player outside $\bar{N}$ has scored against players inside $\bar{N}$, which contradicts the irreducibility of A.

Define $C = (\breve{B})^{-1}$, $\breve{r}^{\text{fb}} = (r_h^{\text{fb}}(A))_{h\in\breve{N}}$, $B^i = (B_{hi})_{h\in\breve{N}}$ and $B^k = (B_{hk})_{h\in\breve{N}}$. Then, using (7.10) we have $\breve{B}\breve{r}^{\text{fb}} = -B^i r_i^{\text{fb}}(A) - B^k r_k^{\text{fb}}(A)$ and hence, $\breve{r}^{\text{fb}} = C(-B^i r_i^{\text{fb}}(A) - B^k r_k^{\text{fb}}(A))$. So, for all $\ell \in \breve{N}$,

$$r_\ell^{\text{fb}} = \breve{r}_\ell^{\text{fb}} = \sum_{h\in\breve{N}} C_{\ell h}(-B_{hi} r_i^{\text{fb}}(A) - B_{hk} r_k^{\text{fb}}(A)).$$

Define $\gamma_\ell^i = -\sum_{h\in\breve{N}} C_{\ell h} B_{hi}$ and $\gamma_\ell^k = -\sum_{h\in\breve{N}} C_{\ell h} B_{hk}$. Then, for each $\ell \in \breve{N}$,

$$r_\ell^{\text{fb}}(A) = \gamma_\ell^i r_i^{\text{fb}}(A) + \gamma_\ell^k r_k^{\text{fb}}(A). \tag{7.11}$$

Furthermore, equation i in $Br^{\text{fb}} = 0$ is

$$B_{ii} r_i^{\text{fb}}(A) + B_{ik} r_k^{\text{fb}}(A) + \sum_{\ell\in\breve{N}} B_{i\ell} r_\ell^{\text{fb}}(A) = 0. \tag{7.12}$$

Define $\Gamma^{i,i} = -\sum_{\ell\in\breve{N}} B_{i\ell}\gamma_\ell^i$ and $\Gamma^{i,k} = -\sum_{\ell\in\breve{N}} B_{i\ell}\gamma_\ell^k$. Then, plugging in the expression of each r_ℓ^{rb} according to (7.11) into (7.12), we get

$$(B_{ii} - \Gamma^{i,i}) r_i^{\text{fb}}(A) + (B_{ik} - \Gamma^{i,k}) r_k^{\text{fb}}(A) = 0. \tag{7.13}$$

Now, adding up (7.11) over all $\ell \in \breve{N}$ and using that $\sum_{h\in N} r_h^{\text{fb}}(A) = 1$,

$$(1 + \sum_{\ell\in\breve{N}} \gamma_\ell^i) r_i^{\text{fb}}(A) + (1 + \sum_{\ell\in\breve{N}} \gamma_\ell^k) r_k^{\text{fb}}(A) = 1. \tag{7.14}$$

Define $\sigma_i = \sum_{\ell\in\breve{N}} \gamma^i_\ell$ and $\sigma_k = \sum_{\ell\in\breve{N}} \gamma^k_\ell$. Then, solving equations (7.13) and (7.14), we get

$$r_i^{\text{fb}}(A) = \frac{1}{(1+\sigma_i)+(1+\sigma_k)(\frac{B_{ii}-\Gamma^{i,i}}{-B_{ik}+\Gamma^{i,k}})} \quad \text{and} \quad r_k^{\text{fb}}(A) = \frac{1}{(1+\sigma_k)+(1+\sigma_i)(\frac{-B_{ik}+\Gamma^{i,k}}{B_{ii}-\Gamma^{i,i}})} \tag{7.15}$$

To understand how $r_i^{\text{fb}}(A)$ and $r_k^{\text{fb}}(A)$ vary with B_{ik} and B_{ii}, it is convenient to know the signs of γ^i and γ^k. We claim that both γ^i and γ^k are nonnegative vectors. By definition, $\gamma^i = -CB^i$ and, since $C^{-1} = \breve{B}$, $\breve{B}\gamma = -B^i$. Furthermore, $-B^i \geq 0$. Because matrix $\breve{B}$ and vectors γ^i and $-B^i$ satisfy the conditions of Lemma 7.6.7, γ^i is nonnegative. The argument for γ^k is analogous using $-B^k$ instead of $-B^i$. Then, because γ^k is nonnegative, also $\Gamma^{i,k}$ is nonnegative and $B_{ik} - \Gamma^{i,k}$ is non-positive. It follows, using (7.13) and the fact that all the components of $r^{\text{fb}}(A)$ are positive, that $B_{ii} - \Gamma^{i,i}$ is non-negative. The nonnegativity of γ^i and γ^k implies that σ_i and σ_k are also nonnegative.

We reexamine now equation (7.15). Note that γ^i, γ^k, $\Gamma^{i,i}$ and $\Gamma^{i,k}$ only depend on matrix $\breve{B}$. Decreasing the performance of player i against player k corresponds with strictly decreasing $-B_{ik}(= A_{ik})$, increasing $-B_{ki}$ by the same amount, and a strict increase of B_{ii}. Therefore, $\frac{B_{ii}-\Gamma^{i,i}}{-B_{ik}+\Gamma^{i,k}}$ strictly increases and so (7.15) shows that $r_i^{\text{fb}}(A)$ strictly decreases and $r_k^{\text{fb}}(A)$ strictly increases.

Finally, we can also combine (7.11) and (7.13) to obtain the expression for $r_\ell^{\text{fb}}(A)$, with $\ell \in \breve{N}$. In this case we get

$$r_\ell^{\text{fb}}(A) = \Big(\gamma^i_\ell + \gamma^k_\ell \frac{B_{ii}-\Gamma^{i,i}}{-B_{ik}+\Gamma^{i,k}}\Big) r_i^{\text{fb}}(A) = \Big(\gamma^k_\ell + \gamma^i_\ell \frac{-B_{ik}+\Gamma^{i,k}}{B_{ii}-\Gamma^{i,i}}\Big) r_k^{\text{fb}}(A).$$

Therefore, for each $\ell \in N\backslash\{i,k\}$, both $\frac{r_\ell^{\text{fb}}(A)}{r_i^{\text{fb}}(A)}$ and $\frac{r_\ell^{\text{fb}}(A)}{r_k^{\text{fb}}(A)}$ weakly decrease. From this, the statement follows. $\square$

7.7 Discussion

Table 7.7.1 summarizes the behavior of the ranking methods we have studied with respect to the different properties. The scores ranking method satisfies several attractive properties such as PRB and SVL. However, a potential drawback is revealed by the property IIM: the scores ranking method only looks at the aggregate score of each player, ignoring the opponents he has faced to obtain this score. All other ranking methods under consideration are responsive to the strength of the opponents of each player. One can argue that IIM is a natural property for round-robin tournaments. Within this class of tournaments the ranking methods that satisfy SCC also satisfy IIM.

Besides the scores ranking, fair bets and recursive Buchholz are the two ranking methods that satisfy the property of NNRB. Note that this is an advantage of fair bets and recursive Buchholz with respect to their 'simple' counterparts Neustadtl and Buchholz. From our point of view a drawback of the fair bets ranking is that it violates SVL, which imposes the natural requirement that if we reverse all the results in the tournament, then the corresponding ranking should be obtained by reverting the original ranking as well. Provided that our conjecture regarding maximum likelihood and NNRB holds, maximum likelihood and recursive Buchholz satisfy the same set of considered properties. One potential advantage of φ^{rb} with respect to φ^{ml} is that, since φ^{ml} requires to solve a system of non-linear equations, it may be very hard to compute in settings where there is a high number of alternatives to be ranked.

	Scores	Neustadtl	Fair bets	Maximum Likelihood	Recursive Performance	Buchholz	Recursive Buchholz
ANO	✓	✓	✓	✓	✓	✓	✓
HOM	✓	✓	✓	✓	✓	✓	✓
SYM	✓	✓	✓	✓	✓	✓	✓
SS	X	X	✓	✓	X	X	✓
FP	✓	X	✓	✓	✓	X	✓
SVL	✓	X	X	✓	✓	✓	✓
NRL	X	X	✓	X	X	X	X
SCC	✓	X	X	✓	✓	✓*	✓
SSCC	✓	X	X	✓	✓	✓*	✓
IIM	✓	X	X	X	X	X	X
PRB	✓	X	X	X	X	X	X
NNRB	✓	X	✓	?	X	X	✓

*Requires $|N| > 2$.

Table 7.7.1: Ranking methods and properties.

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